Compressed Sensing for fusion frames

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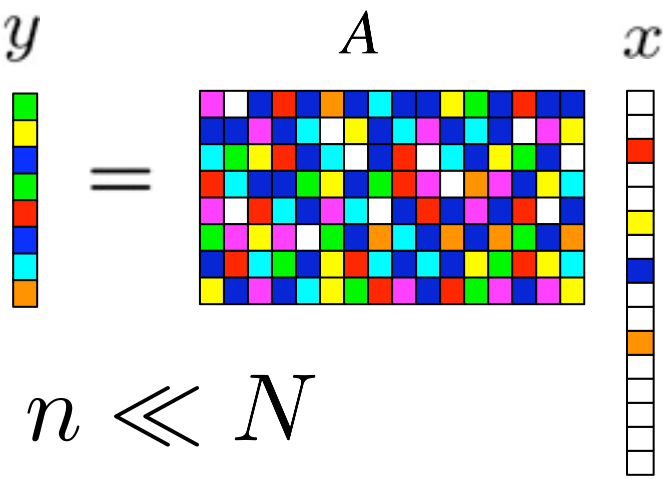
COMPRESSED SENSING

Compressed Sensing Measurement Model

$n \times 1$
measurements



$N \times 1$
sparse
signal

$K < n \ll N$

K
nonzero
entries

- x is K-sparse or K-compressible
- A **random**, satisfies a *restricted isometry property (RIP)*

  A has RIP of order $2K$ with constant δ

  If there exists δ s.t. for all $2K$-sparse x:

  $$(1-\delta)\|\mathbf{x}\|_2^2 \leq \|\mathbf{Ax}\| \leq (1+\delta)\|\mathbf{x}\|_2^2$$

- $n = O(K \log N/K)$
- A also has small *coherence*

$$\mu \triangleq \max_{i \neq j} |\langle \mathbf{a}_i, \mathbf{a}_j \rangle|$$

CS Reconstruction

- Reconstruction using **sparse approximation**:
  - Find sparsest $\mathbf{x}$ such that $\mathbf{y} \approx \Phi\mathbf{x}$
- **Convex optimization** approach:
  - Minimize l_1 norm: e.g.,

$$\widehat{\mathbf{x}} = \arg\max_{\mathbf{x}} \|\mathbf{x}\|_1 \quad \text{s.t. } \mathbf{y} = \mathbf{A}\mathbf{x}$$

- **Greedy algorithms** approach:
  - MP, OMP, ROMP, StOMP, CoSaMP, …
  - PYAMP (Pick Your Acronym Matching Pursuit)
- If coherence μ or RIP δ is ***small***: Exact reconstruction

FUSION FRAMES

Fusion Frame

A collection of subspaces $\{W_j\}, j=1,\ldots,N$ and a set of weights v_j such that there exist universal constants $0<A\leq B<\infty$:

$$A\|\mathbf{x}\|_2^2 \leq \sum_{j=1}^{N} v_j^2 \,\|\mathbf{P}_j(\mathbf{x})\|_2^2 \leq B\|\mathbf{x}\|_2^2, \text{ for all } \mathbf{x} \in \mathbb{R}^M$$

Similar to the definition of a frame:

$$A\|\mathbf{x}\|_2^2 \leq \sum_{j=1}^{N} |\langle \mathbf{f}_j, \mathbf{x}\rangle|_2^2 \leq B\|\mathbf{x}\|_2^2, \text{ for all } \mathbf{x} \in \mathbb{R}^M$$

Extends the concepts of a frame to a

richer, more descriptive representation

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Projection onto W_j

Similar to the definition of a frame:

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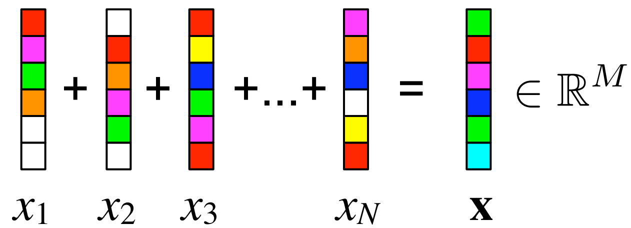
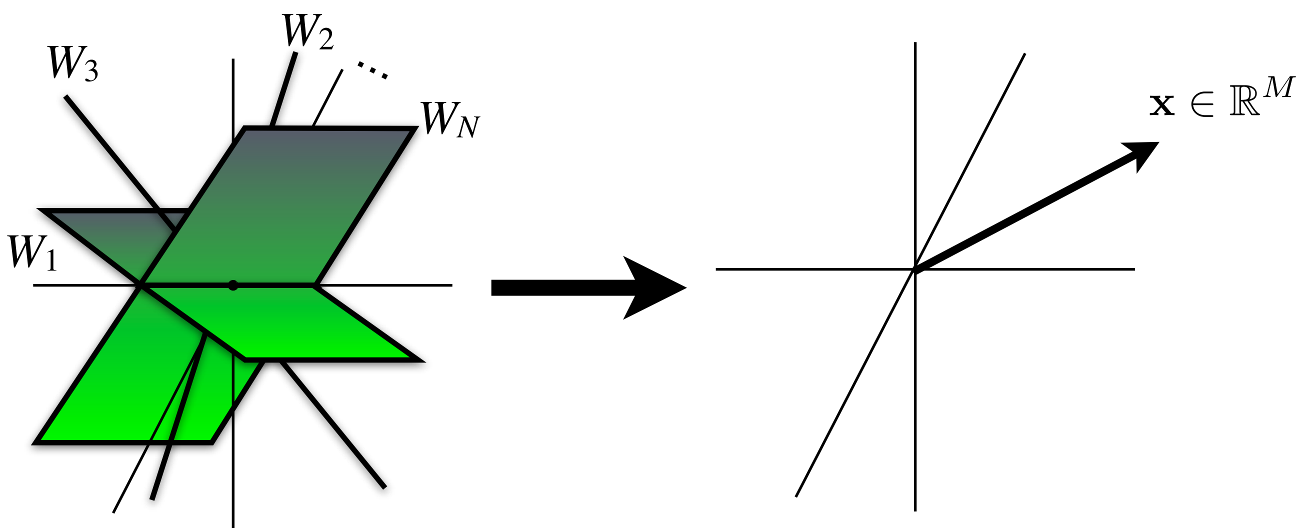
Extends the concepts of a frame to a **richer, more descriptive representation**

Fusion Frame Vectors

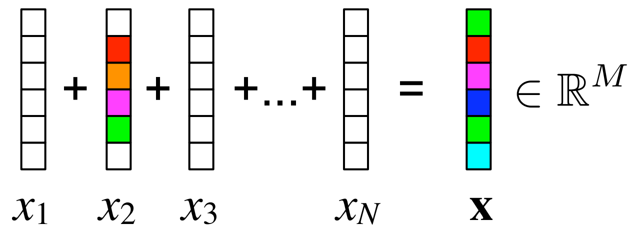


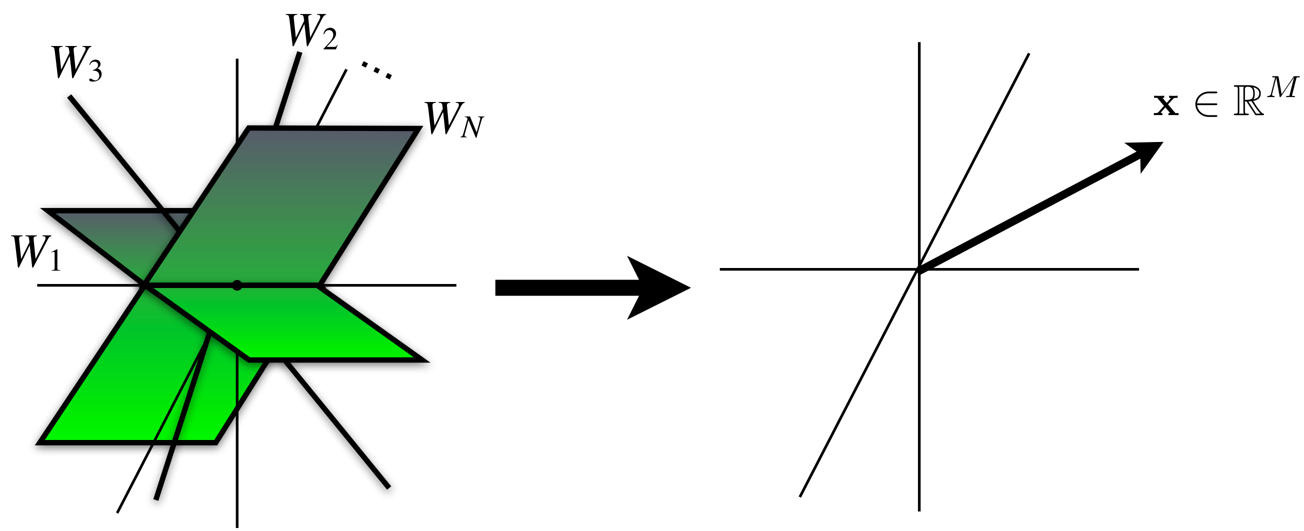


Fusion frame vector: $\mathbf{x} = \sum_{j=1}^{N} v_j x_j, \;\; x_j \in W_j, \;\; j = 1, \ldots, N$

The x_j can be thought of as **vector-valued "coefficients"**

Fusion Frame Sparsity





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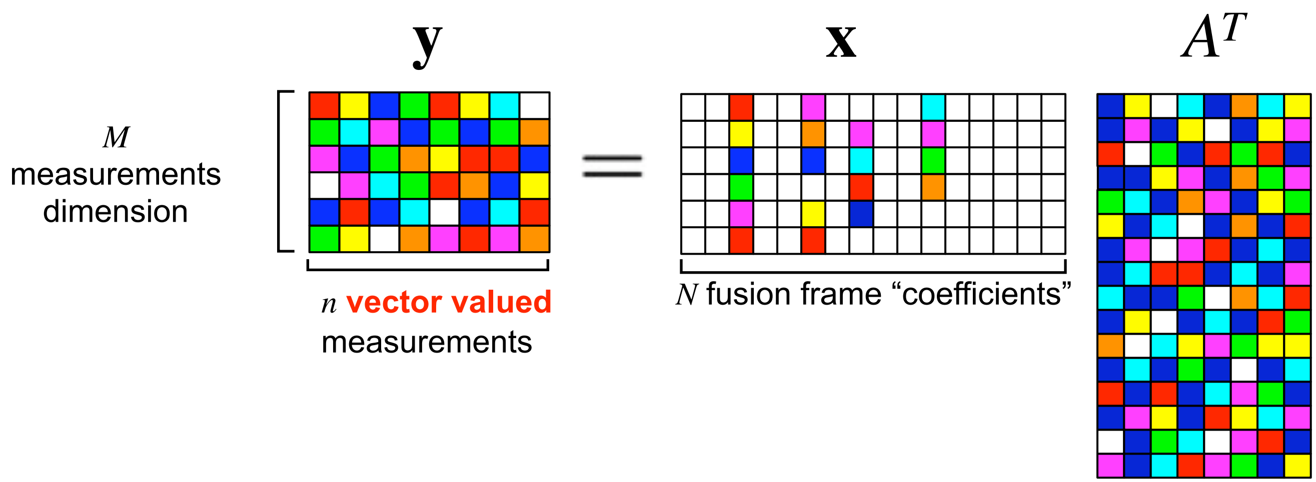
The x_j can be thought of as **vector-valued "coefficients"**

Sparsity: very **few** of the x_j are non-zero

MITSUBISHI ELECTRIC
Changes for the better

COMPRESSED SENSING FOR FUSION FRAMES

Fusion Frame Measurements



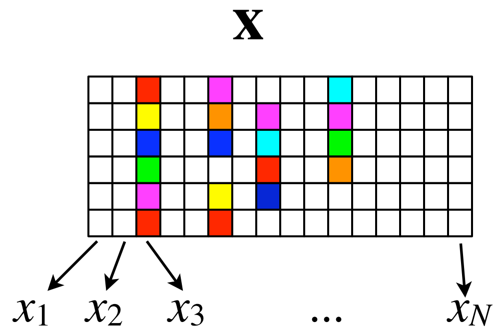
$$\mathbf{y} = \mathbf{x}\mathbf{A}^T$$

$\mathbf{x}$ has a **sparse** fusion frame representation

Can we recover $\mathbf{x}$ using an l_1-type minimization?

But what is l_1 in this case?

Fusion frame l_1 Norm



$$\|\mathbf{x}\|_1 \triangleq \sum_{j=1}^{N} \|x_j\|_2$$

Fusion frame l_1 norm: mixed l_1/l_2 norm of the fusion frame coefficients

Signal Recovery:

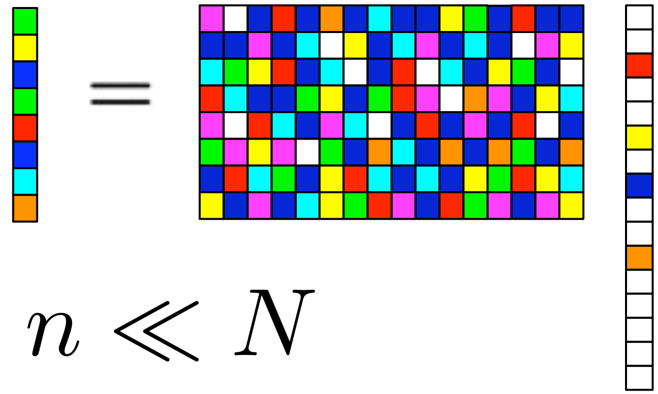
$$\widehat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_1 \text{ s.t. } \mathbf{y} = \mathbf{x}\mathbf{A}^T$$

When does this work?

Compressed Sensing Measurement Model

y A x

$n \times 1$
measurements



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y $\quad$ A $\quad$ x

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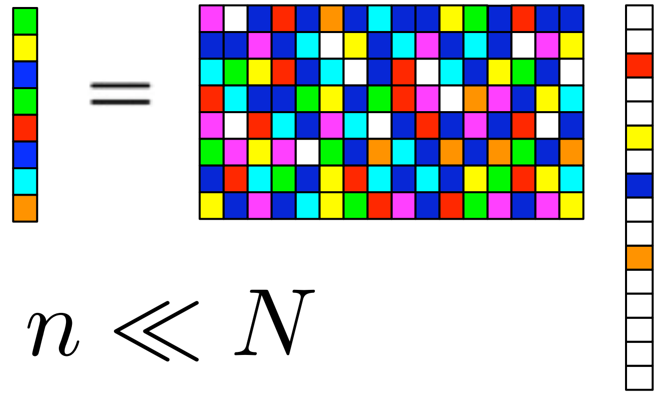

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- n=O(Klog$N$/K)
- A also has small *coherence*

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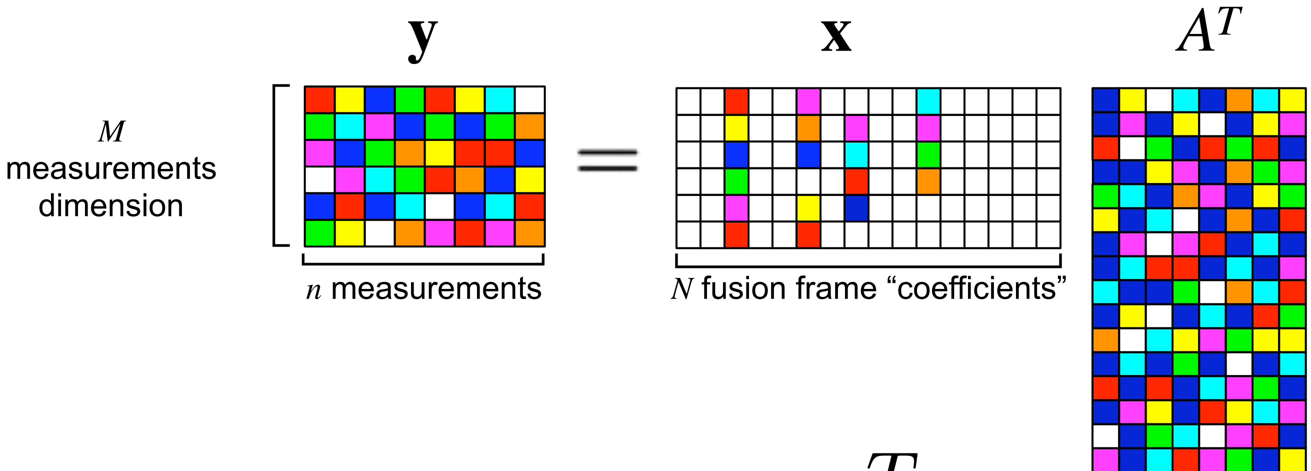
Equivalent for fusion frame measurements?

Restricted Isometry Property



$$\mathbf{y} = \mathbf{x}\mathbf{A}^T$$

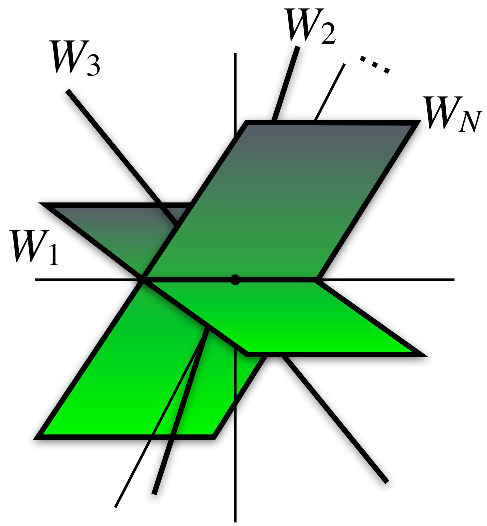
- If A has RIP of order 2K we can still recover vectors with a K-sparse fusion frame representation.
- RIP definition does not change
- Recovery using l_1/l_2 minimization

Fusion l_1 norm (mixed l_1/l_2)

$$\widehat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_1 \text{ s.t. } \mathbf{y} = \mathbf{x}\mathbf{A}^T$$

- RIP definition doesn't change.
- Special structure of the fusion frame not incorporated in the RIP

Fusion Coherence



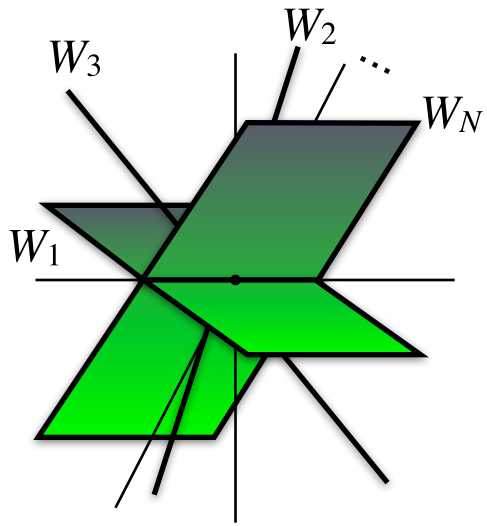
$$\mu_f = \max_{j \neq k} \left[|\langle \mathbf{a}_j, \mathbf{a}_k \rangle| \cdot |\lambda_{\max}(\mathbf{P}_j \mathbf{P}_k)|^{1/2} \right]$$

Reconstruction possible if:

$$K < \frac{1}{2}\left(1 - \mu_f^{-1}\right)$$

- P_j, P_k: Projection onto W_j, W_k
- λ_{max}: Largest eigenvalue of $(P_j P_k)$
  - max cosine of principal angles between subspaces
  - Large angles ↔ small λ_{max} ↔ can have large $|\langle \mathbf{a}_j, \mathbf{a}_k \rangle|$
- Incorporates information about subspace structure

Fusion Coherence



$$\mu_f = \max_{j \neq k} \left[\left| \langle \mathbf{a}_j, \mathbf{a}_k \rangle \right| \cdot \left| \lambda_{\max} \left(\mathbf{P}_j \mathbf{P}_k \right) \right|^{1/2} \right]$$

Standard Definition

Incorporates fusion frame properties

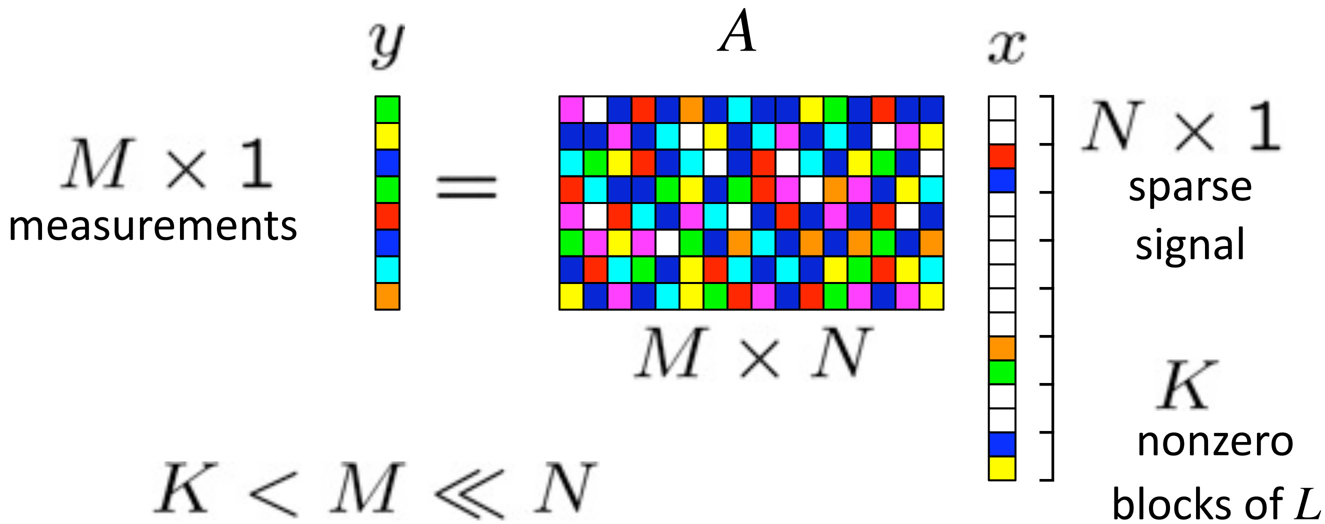
Reconstruction possible if:

$$K < \frac{1}{2}\left(1 - \mu_f^{-1}\right)$$

- P_j, P_k: Projection onto W_j, W_k
- $\lambda_{\max}$: Largest eigenvalue of $(P_j P_k)$
  - max cosine of principal angles between subspaces
  - Large angles $\leftrightarrow$ small $\lambda_{\max}$ $\leftrightarrow$ can have large $|\langle \mathbf{a}_j, \mathbf{a}_k \rangle|$
- Incorporates information about subspace structure

CONNECTIONS

Block Sparsity

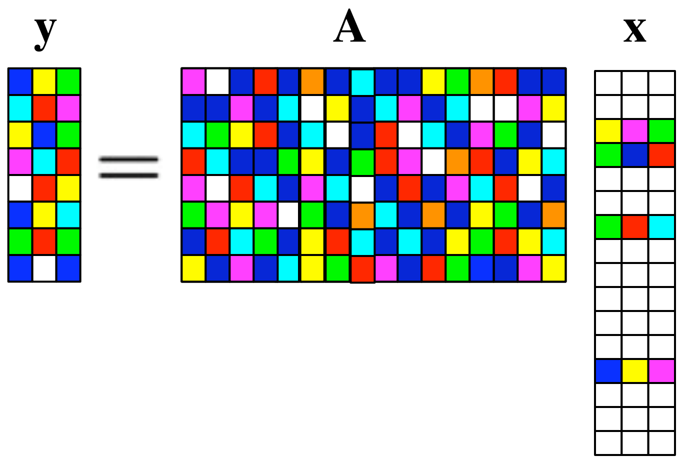


$$K < M \ll N$$

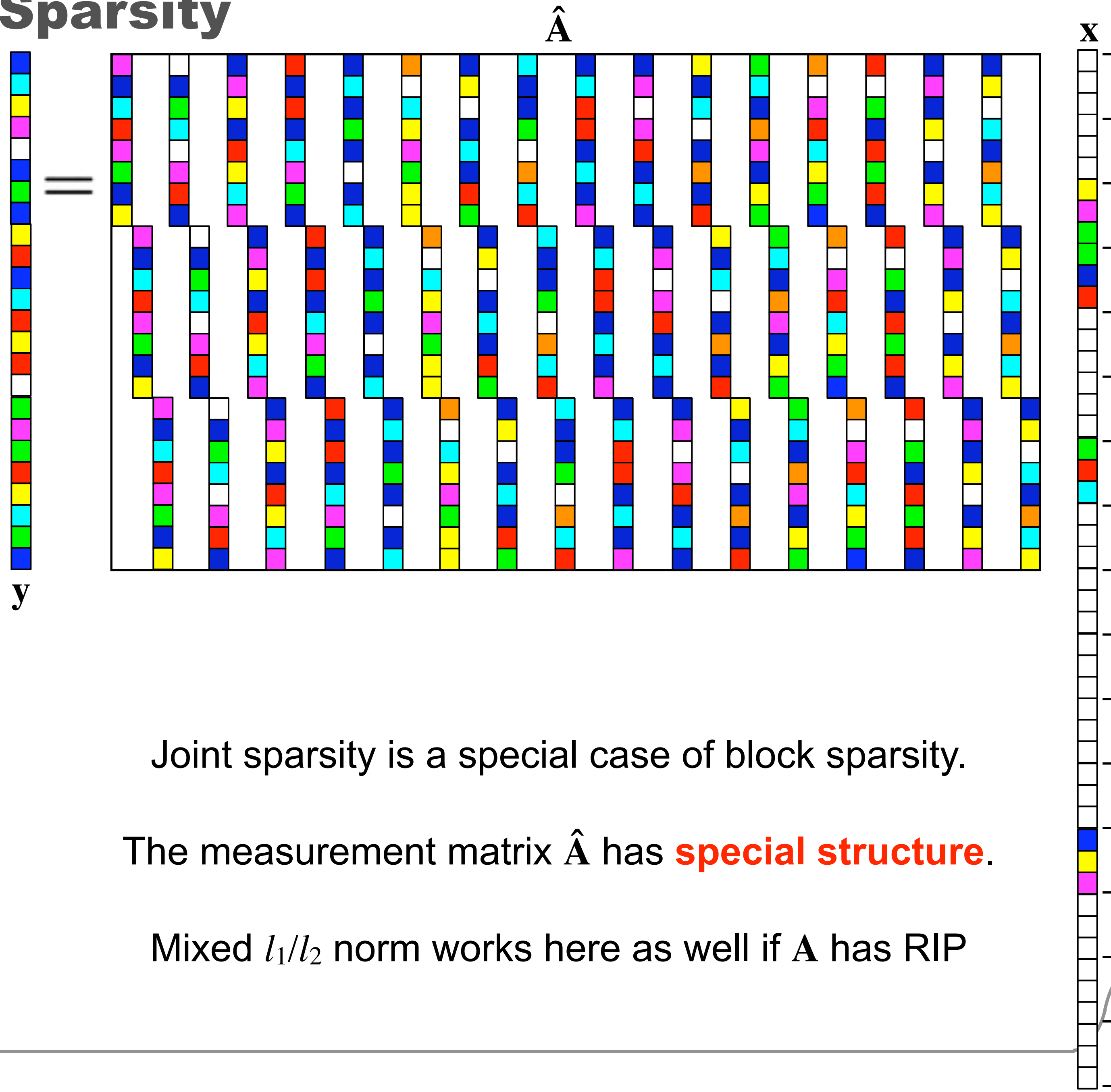
Mixed l_1/l_2 norm known to work and proven if $\mathbf{A}$ has RIP.

Blocks are not allowed to overlap

Joint Sparsity



Joint Sparsity

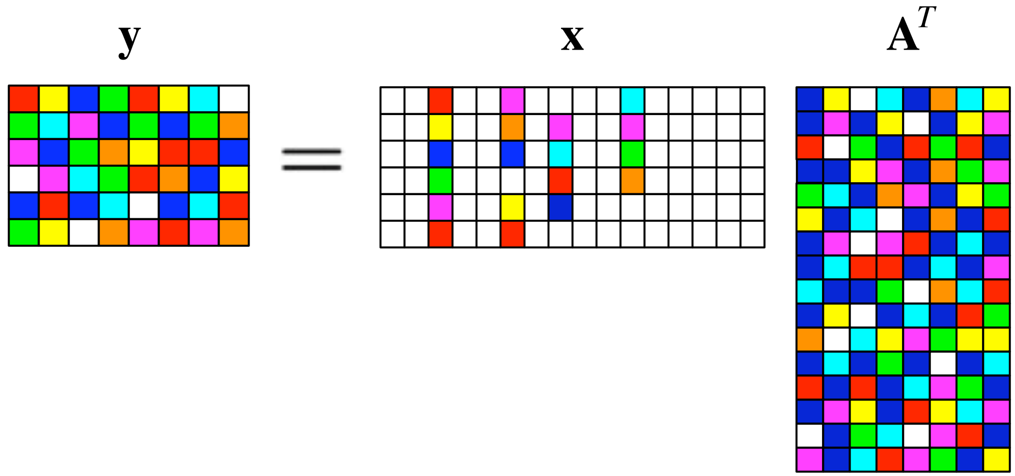


Joint sparsity is a special case of block sparsity.

The measurement matrix $\hat{\mathbf{A}}$ has **special structure**.

Mixed l_1/l_2 norm works here as well if $\mathbf{A}$ has RIP

Fusion Frame Measurements



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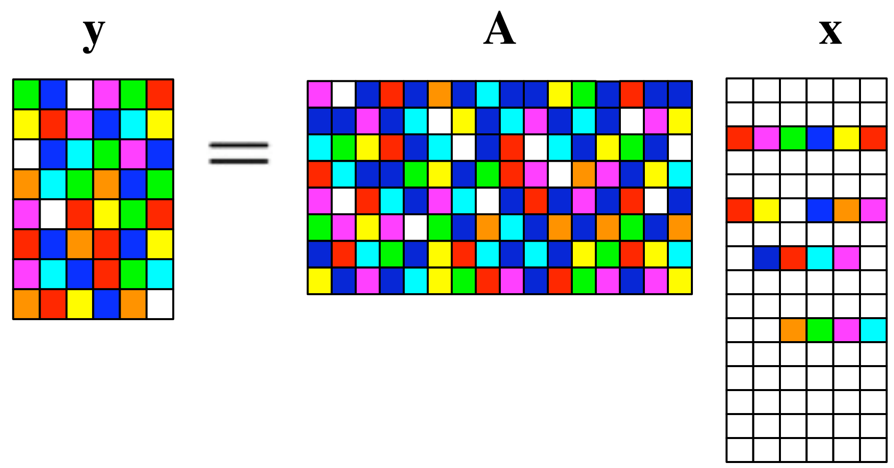


Fusion frame measurements generalize joint sparsity measurements

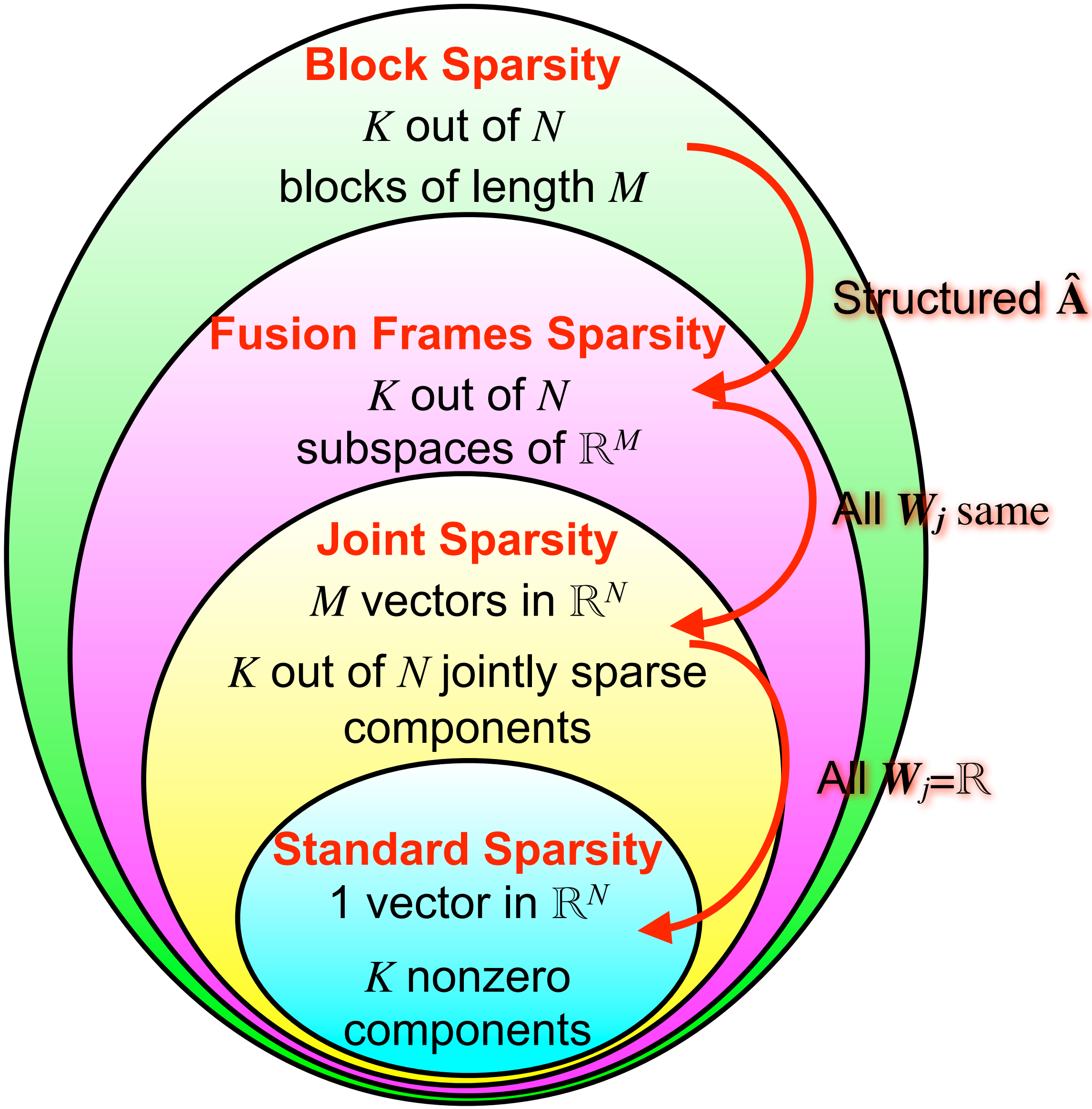
We use extra **information on the subspaces** to relax the requirements on $\mathbf{A}$

If $W_1 = W_2 = \ldots = W_N = \mathbb{R}^M$ we revert to joint sparsity

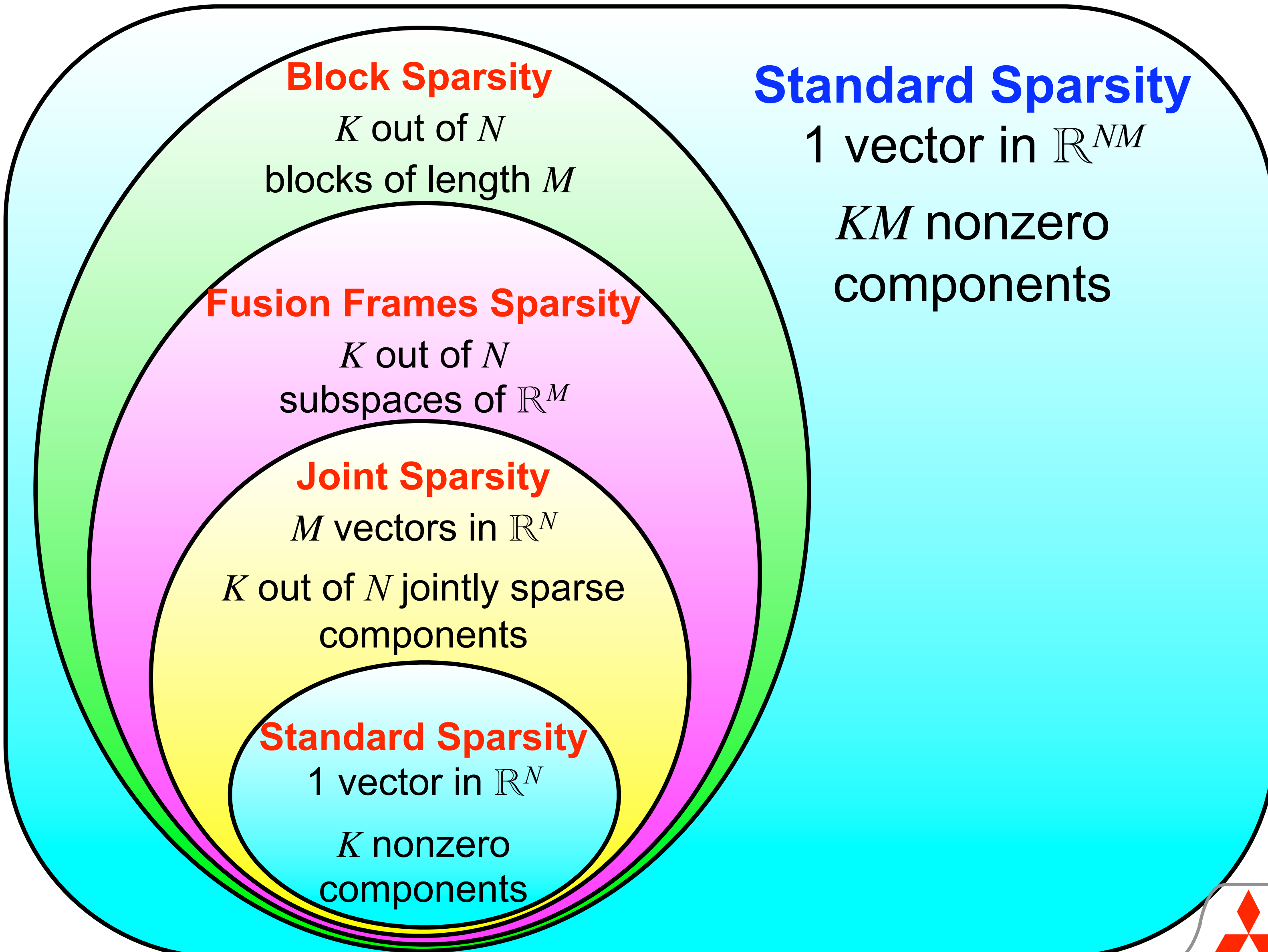
If $W_1 = W_2 = \ldots = W_N = \mathbb{R}$ we revert to standard CS

We are still a special case of block sparsity

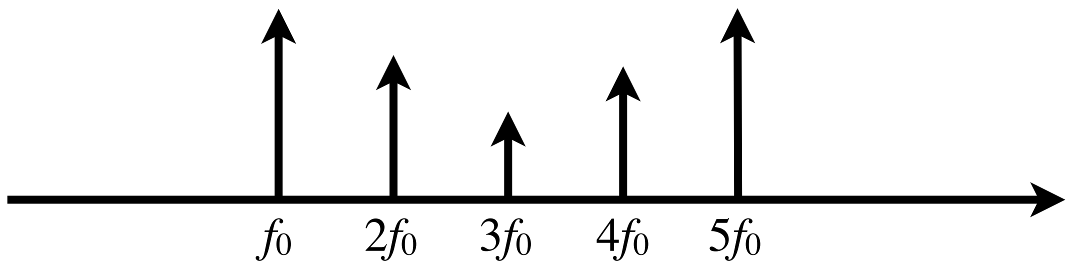
Model Hierarchy



Model Hierarchy



Application/Motivation: Dictionaries of Subspaces



- Targets that span subspaces
  - e.g., harmonics of a fundamental frequency
- The dictionary becomes a collection of subspaces
  - Musical instruments
  - Vehicle identification
- First step for hierarchical identification
  - Once the subspace is identified, further local processing is more efficient

Conclusions

- We extended standard CS results to **fusion frames**
- Using l_1/l_2 norms everything transfers almost **as expected**
- We exploit the **rich structure** of fusion frames
- **Fusion coherence** incorporates this structure
- Still to do: incorporate the structure in **RIP**
- A richer model for **joint sparsity**
- A model for vector based measurements