Compressed Sensing for fusion frames

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COMPRESSED SENSING
Compressed Sensing Measurement Model

\[ y = A x \]

- \( n \times 1 \) measurements
- \( n = O(K \log N/K) \)
- \( A \) has RIP of order \( 2K \) with constant \( \delta \)
- \( \mu \triangleq \max_{i \neq j} |\langle a_i, a_j \rangle| \)
- \( A \) also has small coherence

- \( x \) is \( K \)-sparse or \( K \)-compressible
- \( A \) random, satisfies a restricted isometry property (RIP)

If there exists \( \delta \) s.t. for all \( 2K \)-sparse \( x \):

\[ (1 - \delta)\|x\|_2^2 \leq \|Ax\| \leq (1 + \delta)\|x\|_2^2 \]
CS Reconstruction

• Reconstruction using **sparse approximation**:  
  – Find sparsest \( x \) such that \( y \approx \Phi x \)

• **Convex optimization** approach:  
  – Minimize \( l_1 \) norm: e.g.,  
    \[
    \hat{x} = \arg \max_x ||x||_1 \quad \text{s.t.} \quad y = Ax
    \]

• **Greedy algorithms** approach:  
  – MP, OMP, ROMP, StOMP, CoSaMP, …  
  – PYAMP (Pick Your Acronym Matching Pursuit)

• If coherence \( \mu \) or RIP \( \delta \) is **small**: Exact reconstruction
FUSION FRAMES
Fusion Frame

A collection of subspaces \( \{W_j\}, j=1,\ldots,N \) and a set of weights \( v_j \) such that there exist universal constants \( 0<A\leq B<\infty \):

\[
A\|x\|_2^2 \leq \sum_{j=1}^{N} v_j^2 \|P_j(x)\|_2^2 \leq B\|x\|_2^2, \text{ for all } x \in \mathbb{R}^M
\]

Similar to the definition of a frame:

\[
A\|x\|_2^2 \leq \sum_{j=1}^{N} |\langle f_j, x \rangle|_2^2 \leq B\|x\|_2^2, \text{ for all } x \in \mathbb{R}^M
\]

Extends the concepts of a frame to a richer, more descriptive representation
Fusion Frame

A collection of subspaces \( \{ W_j \}, j=1, \ldots, N \) and a set of weights \( v_j \) such that there exist universal constants \( 0 < A \leq B < \infty \):

\[
A \| x \|_2^2 \leq \sum_{j=1}^{N} v_j^2 \| P_j (x) \|_2^2 \leq B \| x \|_2^2, \quad \text{for all } x \in \mathbb{R}^M
\]

Projection onto \( W_j \)

Similar to the definition of a frame:

\[
A \| x \|_2^2 \leq \sum_{j=1}^{N} | \langle f_j, x \rangle |_2^2 \leq B \| x \|_2^2, \quad \text{for all } x \in \mathbb{R}^M
\]

Extends the concepts of a frame to a richer, more descriptive representation
Fusion Frame Vectors

\[ x_1 + x_2 + x_3 + \ldots + x_N = x \in \mathbb{R}^M \]

Fusion frame vector: 

\[ x = \sum_{j=1}^{N} v_j x_j, \quad x_j \in W_j, \quad j = 1, \ldots, N \]

The \( x_j \) can be thought of as vector-valued “coefficients”
Fusion Frame Sparsity

Fusion frame vector: \[ \mathbf{x} = \sum_{j=1}^{N} v_j x_j, \quad x_j \in W_j, \quad j = 1, \ldots, N \]

The \( x_j \) can be thought of as vector-valued “coefficients”

Sparsity: very few of the \( x_j \) are non-zero
COMPRESSED SENSING FOR FUSION FRAMES
Fusion Frame Measurements

\[ y = xA^T \]

- \( y \) has \( M \) measurements dimension
- \( x \) has a **sparse** fusion frame representation
- \( A^T \) has \( N \) fusion frame “coefficients”

Can we recover \( x \) using an \( l_1 \)-type minimization?

But what is \( l_1 \) in this case?
Fusion frame $l_1$ Norm

Fusion frame $l_1$ norm: mixed $l_1/l_2$ norm of the fusion frame coefficients

Signal Recovery:

$$\hat{x} = \arg\min_x \|x\|_1 \text{ s.t. } y = xA^T$$

When does this work?
Compressed Sensing Measurement Model

\[ n \times 1 \text{ measurements} = A \quad K < n \ll N \quad \text{sparse signal} \]

- \( x \) is \( K \)-sparse or \( K \)-compressible
- \( A \) random, satisfies a restricted isometry property (RIP)
  - \( A \) has RIP of order \( 2K \) with constant \( \delta \)
  - If there exists \( \delta \) s.t. for all \( 2K \)-sparse \( x \):
    \[ (1 - \delta)\|x\|_2^2 \leq \|Ax\| \leq (1 + \delta)\|x\|_2^2 \]
- \( n = O(K \log N / K) \)
- \( A \) also has small coherence

\[ \mu \triangleq \max_{i \neq j} |\langle a_i, a_j \rangle| \]
Compressed Sensing Measurement Model

\[ \begin{align*}
\mathbf{y} &= \mathbf{A} \mathbf{x} \\
n \times 1 & \quad \text{measurements} \\
K & \leq n \ll N \\
N \times 1 & \quad \text{sparse \ signal} \\
K & \quad \text{nonzero \ entries}
\end{align*} \]

- \( \mathbf{x} \) is \( K \)-sparse or \( K \)-compressible
- \( \mathbf{A} \) random, satisfies a restricted isometry property (RIP)
- \( n = O(K \log N / K) \)
- \( \mathbf{A} \) also has small coherence

\[ \mu \triangleq \max_{i \neq j} |\langle \mathbf{a}_i, \mathbf{a}_j \rangle| \]

Equivalent for fusion frame measurements?
Restricted Isometry Property

If $A$ has RIP of order $2K$ we can still recover vectors with a $K$-sparse fusion frame representation.

- RIP definition does not change
- Recovery using $l_1/l_2$ minimization

$$\hat{x} = \arg\min_x \|x\|_1 \text{ s.t. } y = xA^T$$

- RIP definition doesn’t change.
- Special structure of the fusion frame not incorporated in the RIP

Fusion $l_1$ norm (mixed $l_1/l_2$)
Fusion Coherence

\[
\mu_f = \max_{j \neq k} \left[ |\langle a_j, a_k \rangle| \cdot |\lambda_{\text{max}} (P_j P_k)|^{1/2} \right]
\]

Reconstruction possible if:

\[
K < \frac{1}{2} \left( 1 - \mu_f^{-1} \right)
\]

- \(P_j, P_k\): Projection onto \(W_j, W_k\)

- \(\lambda_{\text{max}}\): Largest eigenvalue of \((P_j P_k)\)
  - max cosine of principal angles between subspaces
  - Large angles \(\leftrightarrow\) small \(\lambda_{\text{max}}\) \(\leftrightarrow\) can have large \(|\langle a_j, a_k \rangle|\)

- Incorporates information about subspace structure
Fusion Coherence

\[ \mu_f = \max_{j \neq k} \left[ \langle a_j, a_k \rangle \cdot |\lambda_{\text{max}}(P_j P_k)|^{1/2} \right] \]

Standard Definition

Incorporates fusion frame properties

Reconstruction possible if:

\[ K < \frac{1}{2} \left( 1 - \mu_f^{-1} \right) \]

- **\( P_j, P_k \): Projection onto \( W_j, W_k \)**
- **\( \lambda_{\text{max}} \): Largest eigenvalue of \( (P_j P_k) \)**
  - max cosine of principal angles between subspaces
  - Large angles \( \leftrightarrow \) small \( \lambda_{\text{max}} \) \( \leftrightarrow \) can have large \( |\langle a_j, a_k \rangle|\)
- **Incorporates information about subspace structure**
Block Sparsity

Mixed $l_1/l_2$ norm known to work and proven if $A$ has RIP.

Blocks are not allowed to overlap
Joint Sparsity

\[ y = A x \]
Joint sparsity is a special case of block sparsity.

The measurement matrix $\hat{A}$ has special structure.

Mixed $l_1/l_2$ norm works here as well if $A$ has RIP.
Fusion Frame Measurements

\[ y = x A^T \]
Fusion Frame Measurements

Fusion frame measurements generalize joint sparsity measurements

We use extra information on the subspaces to relax the requirements on $A$

If $W_1=W_2=\ldots=W_N=\mathbb{R}^M$ we revert to joint sparsity

If $W_1=W_2=\ldots=W_N=\mathbb{R}$ we revert to standard CS

We are still a special case of block sparsity
Model Hierarchy

Block Sparsity
$K$ out of $N$
blocs of length $M$

Fusion Frames Sparsity
$K$ out of $N$
subspaces of $\mathbb{R}^M$

Joint Sparsity
$M$ vectors in $\mathbb{R}^N$
$K$ out of $N$ jointly sparse
components

Standard Sparsity
1 vector in $\mathbb{R}^N$
$K$ nonzero
components

Structured $\hat{A}$

All $W_j$ same

All $W_j = \mathbb{R}$
Model Hierarchy

- **Standard Sparsity**
  - 1 vector in $\mathbb{R}^{NM}$
  - $KM$ nonzero components

- **Joint Sparsity**
  - $M$ vectors in $\mathbb{R}^N$
  - $K$ out of $N$ jointly sparse components

- **Fusion Frames Sparsity**
  - $K$ out of $N$ subspaces of $\mathbb{R}^M$

- **Block Sparsity**
  - $K$ out of $N$ blocks of length $M$
Application/Motivation: Dictionaries of Subspaces

• Targets that span subspaces
  – e.g., harmonics of a fundamental frequency

• The dictionary becomes a collection of subspaces
  – Musical instruments
  – Vehicle identification

• First step for hierarchical identification
  – Once the subspace is identified, further local processing is more efficient
Conclusions

• We extended standard CS results to fusion frames

• Using $l_1/l_2$ norms everything transfers almost as expected

• We exploit the rich structure of fusion frames

• Fusion coherence incorporates this structure

• Still to do: incorporate the structure in RIP

• A richer model for joint sparsity

• A model for vector based measurements