1-bit Compressive Sensing

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RICE
Compressive Sensing

1. Signal Model
2. Random linear measurements
3. Non-linear reconstruction
Signal Models

Classical Model: Signal lies in a linear vector space (e.g. bandlimited functions)

Sparse Model: Signals of interest are often sparse or compressible

i.e., very few large coefficients, many close to zero.
Sparse Signal Models

Sparse signals have few non-zero coefficients.

Compressible signals have few significant coefficients. The coefficients decay as a power law.

Compressible (\(\ell_p\) ball, \(p<1\))
Compressive Sensing in a Nutshell

If a signal is **sparse**, do not waste effort sampling the **empty space**.

Instead, use fewer samples and allow **ambiguity**.

Use the **sparsity model** to reconstruct and **uniquely resolve** the ambiguity.
Compressive Measurements

\[ y = \Phi x \]
\[ y_i = \langle \phi_i, x \rangle \]

Measurement (Projection)

Reconstruction

\[ \Phi \text{ has rank } M \ll N \]
\[ \Phi \text{ is usually random w/ } M = O(K \log N/K) \]

\[ N = \text{Signal dimensionality} \quad M = \text{Number of measurements} \]
\[ K = \text{Signal sparsity} \quad (\text{dimensionality of } y) \]

\[ N \gg M \geq K \]
Reconstruction should be:

1. **Consistent with measurements:**
   \[ y = \Phi x \]

2. **Consistent with the model:**
   \[ x \text{ is as sparse as possible} \]

**Ambiguity**

\[ \text{null}\{\Phi\} + x \]

**Sparsity measure**

\[ \min_x \|x\|_0 \text{ subject to } y = \Phi x \]

Expensive!
Compressive Sensing

1. Signal Model
2. Random linear measurements
3. Non-linear reconstruction
Beyond Linear Measurements: 1-bit Quantization
Q: Can we quantize measurements to 1-bit:

\[ y = \text{sign}(\Phi x) \]
\[ y_i = \text{sign}(\langle \phi_i, x \rangle) \]

and recover the signal (within a positive scaling factor)?

1-bit measurements are inexpensive.

Focus on bits rather than measurements.

Exact recovery is not possible.
Reconstruction from 1-bit Measurements

Sign information from 1-bit measurements:

\[ y_i = \text{sign}(\Phi x)_i \iff y_i \cdot (\Phi x)_i \geq 0 \]

Reconstruction should enforce model.
Reconstruction should be consistent with measurements.
Reconstruction should enforce a non-trivial solution.

\[ \hat{x} = \arg \min_x \|x\|_1 \]
subject to \[ y_i \cdot (\Phi x)_i \geq 0 \]
and \[ \|x\|_2 = 1 \]
Constraint Relaxation

$$\hat{x} = \arg \min_{x} \|x\|_1$$

subject to  $$y_i \cdot (\Phi x)_i \geq 0$$

and  $$\|x\|_2 = 1$$

We relax the inequality constraints:

$$\hat{x} = \arg \min_{x} \|x\|_1 + \frac{\lambda}{2} \sum_i f (y_i \cdot (\Phi x))$$

subject to  $$\|x\|_2 = 1$$

where $$f(x)$$ is a one sided quadratic:

$$f(x) = \begin{cases} 
x^2 & x \leq 0 \\
0 & x > 0
\end{cases}$$
Fixed point equilibrium

\[ \hat{x} = \arg\min_x \|x\|_1 + \frac{\lambda}{2} \sum_i f(y_i \cdot (\Phi x)) \]

subject to \( \|x\|_2 = 1 \)

**Unconstrained minimization:**

\[ Y \equiv \text{diag}(y) \]

\[ \text{Cost}(x) = g(x) + \frac{\lambda}{2} f(Y \Phi x) \]

\[ \text{Cost}'(x) = g'(x) + \frac{\lambda}{2} (Y \Phi)^T f(Y \Phi x) \]

\[ (g'(x))_i = \begin{cases} -1 & x_i < 0 \\ [-1, 1] & x_i = 0 \\ +1 & x_i > 0 \end{cases} \]

and

\[ \left( \frac{f'(x)}{2} \right)_i = \begin{cases} -x_i & x_i \leq 0 \\ 0 & x_i > 0 \end{cases} \]

No change if gradients are projected on unit sphere.
Minimization algorithm

Big Picture: Gradient descent until equilibrium.

Initialization parameters: \( \hat{x}, \tau \)

1. **Compute** quadratic gradient: \( h = (Y\Phi)^T f'(Y\Phi x) \)

2. **Project** onto sphere: \( h_p = h - \langle \hat{x}, h \rangle \)

3. **Quadratic gradient descent**: \( \hat{x} \leftarrow \hat{x} - \tau h_p \)

4. **Shrink** (\( \ell_1 \) gradient descent):
   \[
   \hat{x}_i \leftarrow \text{sign}(\hat{x}_i) \max \left\{ |\hat{x}_i| - \frac{\tau}{\lambda}, 0 \right\}
   \]

5. **Normalize**: \( \hat{x} \leftarrow \frac{\hat{x}}{||\hat{x}||} \)

6. **Iterate** until equilibrium.
Reconstruction Error ($N=512$)

(a) $K=16$

(b) $K=32$

(c) $K=64$

(d) $K=128$
If the signal is an image, we have more information! (i.e., a better signal model)

Images are sparse in wavelets and positive:

\[ x = W\alpha \]
\[ x_i \geq 0 \]
and \( \alpha \) is sparse

Incorporate better model in the reconstruction:

\[ \hat{\alpha} = \arg\min_\alpha \|\alpha\|_1 \]
subject to \( y_i \times (\Phi W)_i \geq 0 \)
and \( (W\alpha)_i \geq 0 \)
and \( \|\alpha\|_2 = 1 \)
Results

Reconstruction on unit sphere
1 bit per pixel

Classical Compressive Sensing, 1 bit per pixel

Original Image
4096 pixels
256 levels

4096 measurements
1 bit per measurement
Results

Original Image
4096 pixels
256 levels

512 measurements
8 bits per measurement

4096 measurements
1 bit per measurement
4096 bits (1 bit per pixel)

Reconstruction on unit sphere

Classical Compressive Sensing

Reconstruction on unit sphere

512 measurements
1 bit per measurement
512 bits (0.125 bits per pixel)
Q: Can we quantize measurements to 1-bit:

\[ y = \text{sign}(\Phi x) \]
\[ y_i = \text{sign}(\langle \phi_i, x \rangle) \]

and recover the signal (within a positive scaling factor)?

**YES:**

- 1-bit measurements only provide sign information
- We treat measurements as constraints
- We do not try to recover amplitude information
- We enforce reconstruction on the unit sphere
- Better signal model provides dramatic improvements