Recent Advances in Signal Acquisition, Sensing, and Quantization

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Evolution of Sensing

~1930s
Evolution of Sensing

~1930s

Digital Sensing

Today
Evolution of Sensing

~1930s

Digital Sensing

Today

Computational Sensing

Future
Sensing by Sampling

- Long-established paradigm for digital data acquisition
  - *sample* data (A-to-D converter, digital camera, …)
  - *compress* data (signal-dependent, nonlinear)
  - *bottleneck* to performance of modern acquisition systems
Compressive Sensing (CS) [Candés, Romberg, Tao; Donoho]

- New signal acquisition method
  - Samples and compresses in **one simple step**
  - Uses **computation** to reconstruct signal
CS AT A GLANCE
Compressed Sensing Measurement Model

- \( x \) is \( K \)-sparse or \( K \)-compressible
- \( A \) random, satisfies a restricted isometry property (RIP)
  
  \[ A \text{ has RIP of order } 2K \text{ with constant } \delta \]

  If there exists \( \delta \) s.t. for all \( 2K \)-sparse \( x \):
  
  \[ (1 - \delta)\|x\|_2^2 \leq \|Ax\| \leq (1 + \delta)\|x\|_2^2 \]

- \( M = O(K \log N/K) \)
- \( A \) also has small coherence
  
  \[ \mu \triangleq \max_{i \neq j} |\langle a_i, a_j \rangle| \]
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CS RECONSTRUCTION
CS Reconstruction

• Reconstruction using **sparse approximation**:  
  – Find sparsest \( \mathbf{x} \) such that \( \mathbf{y} \approx \mathbf{A}\mathbf{x} \)

• **Convex optimization** approach:  
  – Minimize \( l_1 \) norm: e.g.,  
    \[
    \hat{\mathbf{x}} = \arg \max_{\mathbf{x}} \| \mathbf{x} \|_1 \quad \text{s.t.} \quad \mathbf{y} = \mathbf{A}\mathbf{x}
    \]

• **Greedy algorithms** approach:  
  – MP, OMP, ROMP, StOMP, CoSaMP, …  
  – **AndrewMP**, PYAMP (Pick Your Acronym Matching Pursuit)

• If coherence \( \mu \) or RIP \( \delta \) is **small**: Exact reconstruction
Optimization (Basis Pursuit)

Sparse approximation:

Minimize non-zeros in representation
s.t.: representation is close to signal
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Sparse approximation:

Minimize non-zeros in representation
s.t.: representation is close to signal

\[ \min \| x \|_0 \quad \text{s.t.} \quad y \approx Ax \]
Optimization (Basis Pursuit)

Sparse approximation:

\[
\text{Minimize} \ \| x \|_0 \quad \text{s.t.} \quad y \approx Ax
\]

- **Minimize** non-zeros in representation
- s.t.: representation is close to signal

Number of non-zeros
(sparisity measure)

Data Fidelity
(approximation quality)
Optimization (Basis Pursuit)

Sparse approximation:

\[ \text{Minimize} \ \| x \|_0 \ \text{subject to} \ y \approx Ax \]

- **Number of non-zeros** (sparsity measure)
- **Data Fidelity** (approximation quality)

Combinatorial complexity. Very hard problem!
Optimization (Basis Pursuit)

**Sparse approximation:**

**Minimize** non-zeros in representation
s.t.: representation is close to signal

\[
\begin{align*}
\min \| x \|_0 & \quad \text{s.t. } y \approx Ax \\
\min \| x \|_1 & \quad \text{s.t. } y \approx Ax
\end{align*}
\]

Convex Relaxation
Optimization (Basis Pursuit)

Sparse approximation:

Minimize non-zeros in representation
s.t.: representation is close to signal

$$\min \| x \|_0 \quad \text{s.t.} \quad y \approx Ax$$

Convex Relaxation

$$\min \| x \|_1 \quad \text{s.t.} \quad y \approx Ax$$

Polynomial complexity.
Solved using linear programming.
Why $l_1$ relaxation works

$$\min \| x \|_1 \quad \text{s.t.} \quad y \approx Ax$$

$l_1$ "ball"

Sparse solution

$y = Ax$
Why $l_1$ relaxation works

$$\min ||x||_1 \text{ s.t. } y \approx Ax$$

$l_1$ “ball”

Sparse solution

$y = Ax$
**Greedy Pursuits Core Idea**

- $y$ highly correlated with $A$ at locations where $x$ is high
- $A^T y$ provides a good idea of these locations
  - This is why low coherence is important
    \[
    \mu \triangleq \max_{i \neq j} |\langle a_i, a_j \rangle|
    \]
  - $A^T y$ referred to as *proxy* for $x$
- **General Strategy:**
  - Identify locations
  - Invert the system only on those locations
CoSaMP (Compressive Sampling MP) [Needell and Tropp]

Correlate residual with dictionary → signal proxy

\[ \langle a_k, r \rangle = p_k \]

Signal estimate: \( \hat{x} \)
Support estimate: \( T \)
Residual: \( r = y - A\hat{x} \)
CoSaMP (Compressive Sampling MP) [Needell and Tropp]

Correlate residual with dictionary \( \rightarrow \) signal proxy

\[
\langle a_k, r \rangle = p_k
\]

Select location of largest 2K correlations

\[ \text{supp}(p|_{2K}) \]

Signal estimate: \( \hat{x} \)
Support estimate: \( T \)
Residual: \( r = y - A\hat{x} \)
**CoSaMP (Compressive Sampling MP)** [Needell and Tropp]

Correlate residual with dictionary → signal proxy

\[ \langle a_k, r \rangle = p_k \]

Select location of largest 2K correlations

\[ \Omega = \text{supp}(p|_{2K}) \cup T \]

Add to support set

Invert over support

\[ b = A_{\Omega}^\dagger y \]

Signal estimate: \( \hat{x} \)

Support estimate: \( T \)

Residual: \( r = y - A\hat{x} \)

Add to support set

Truncate and compute residual

\[ T = \text{supp}(b|_K) \]

\[ \hat{x} = b|_K \]

\[ r \leftarrow y - A\hat{x} \]
CoSaMP (Compressive Sampling MP) [Needell and Tropp]

- Correlate residual with dictionary → signal proxy
- Select location of largest $2K$ correlations
- Add to support set: $\Omega = \text{supp}(p|_{2K}) \cup T$
- Invert over support: $b = A^\dagger_\Omega y$
- Truncate and compute residual: $T = \text{supp}(b|_K)$
- Signal estimate: $\hat{x} = b|_K$
- Support estimate: $T$
- Residual: $r = y - A\hat{x}$

Iterate using residual

$$\langle a_k, r \rangle = p_k$$
QUANTIZATION
Classical Sampling and Quantization

**Sampling**: discretization in \textit{time}  
\textbf{Lossless} at the Nyquist rate

**Quantization**: discretization in \textit{amplitude}  
\textbf{Always lossy}

Need \textbf{both} for digital data acquisition
Quantizer Design

- Finite Range Quantizer
  - Finite range necessary
  - Hardware constraints
  - Bit-length constraints

- Design parameters:
  - $G$: Saturation level (threshold)
  - $\Delta$: Quantization interval
  - $B$: Bit-rate (per coefficient)

- Design goals:
  - Smaller interval $\Delta$: Less quantization error
  - Larger threshold $G$: Less saturation error
  - Trade-off: $\Delta = 2^{-BG}$
Classical Quantizer Design

• Linear reconstruction penalizes saturation significantly

• Universal design principle: reduce or eliminate saturation

Given: bit budget $B$ bits/sample
Given: maximum signal amplitude $A$
Set: threshold $G \leq A$, i.e., $\Delta \geq 2^{-B+1}A$

• Drawback: amplitude $A$ often unknown
  – In practice: Automatic Gain Control
  – Setting can be too conservative
Given: **Bit budget** $B$ bits/sample, **Signal norm** $\|x\|_2$

Set quantization **threshold** $G$
- Implicitly sets quantization interval $\Delta = 2^{-B+1}G$
- Implicitly sets saturation rate at $2Q(G/\|x\|_2)$

Classical heuristic: Set $G$ large (avoid saturation)

Note:
- equivalent to fixing $G$ and varying signal amplification
- $Q(\cdot)$ denotes the tail of the Gaussian distribution
Given: **Bit budget** $B$ bits/sample, **Signal norm** $||x||_2$

Set quantization **threshold** $G$
- Implicitly sets quantization **interval** $\Delta = 2^{-B+1}G$
- Implicitly sets **saturation rate** at $2Q(G/||x||_2)$

**Classical heuristic:** Set $G$ large (avoid saturation)

Wrong! Will revisit!

Note:
- equivalent to fixing $G$ and varying signal amplification
- $Q(\cdot)$ denotes the tail of the Gaussian distribution
Reconstruction

- Convex optimization:
  \[
  \hat{x} = \underset{x}{\arg \min} \|x\|_1 \text{ s.t. } \|\tilde{y} - \tilde{A}x\|_2 \leq \epsilon
  \]

- \(\epsilon\) bounds the quantization error

- Question: What should \(\epsilon\) be set to?
  - “Straightforward” if no saturation
    - upper bound on quantization error
    - use a quantization noise model
    - \(l_p, p>2\) constraints instead [e.g. Jaques et al. ’09, others]
  - Saturation can make error large or unbounded

- How to deal with saturated measurements?
  - Treat them as **any other measurement**
  - Treat them as missing data and **ignore them**
  - Treat them as providing **different information**
Dropping Measurements

\[ y = A x \]

Saturated measurements
Dropping Measurements

- Measurements are **democratic** [Davenport, Laska, Boufounos, Baraniuk]
  - They are all equally important
  - We can **drop** the **saturated** ones

- Run the standard optimization on the remaining ones

\[
\hat{x} = \arg\min_x \|x\|_1 \text{ s.t. } \|\tilde{y} - \tilde{A}x\|_2 \leq \epsilon
\]

- The remaining \( \tilde{A} \) still satisfies RIP (as long as we don’t drop too many)
Quantization Trade-Off

- **Increasing** saturation level $G$:
  - We *keep more measurements*, i.e., better reconstruction
  - We have *larger quantization interval*, i.e., worse reconstruction

- **Decreasing** saturation level $G$:
  - We *drop more measurements*, i.e., worse reconstruction
  - We have *smaller quantization interval*, i.e., better reconstruction

- A trade-off exists, *contrary* to the classical heuristics
Experimental Results

Conventional approach (treat saturated data as measurements)
Experimental Results

New approach (discard saturated measurements)

Performance improves
Best performance when saturation occurs
Experimental Results

Side by side comparison
Alternative: Exploit Implicit Information

\[
\hat{x} = \arg \min_x \|x\|_1 \\
\text{s.t. } \|\tilde{y} - \tilde{\Phi}x\|_2 \leq \epsilon \\
\Phi^+ x \geq G - \Delta / 2 \\
\Phi^- x \leq -G + \Delta / 2
\]

Saturation provides information: The measurement is larger than \(G\).

Treat measurement as a constraint!
Experimental Results

Graph showing the relationship between SNR and Threshold (G) for different methods: 
- Saturation Constraint (solid green line)
- Saturation Rejection (dashed red line)
- BPDN (dashed blue line)

The dotted line represents the Average Saturation Rate.
Experimental Results

Note: optimal performance requires 10% saturation
Experimental Results

**Maximum Achievable SNR**

**Range of Improved Performance**

**Consistent Reconstruction:**
- Only *slightly better* SNR performance
- More *robust* to saturation variability
1-BIT COMPRESSED SENSING
1-bit Compressive Sensing

Q: Can we quantize the measurements to 1-bit?

\[ y = \text{sign}(Ax) \]
and recover using a greedy algorithm?

**Motivation:**
1-bit quantization is easy, fast and cheap

**Key issues:**
Need to identify the cost function
Can only recover within a positive scaling factor

\[ \text{sign}(Ax) = \text{sign}(Acx), \quad c \geq 0 \]
1-Bit Information
1-Bit Information
1-Bit Information
1-Bit Information
1-Bit Information
1-Bit Information
1-Bit Information
1-Bit Information
1-Bit Information
1-bit CS Cost Function

- Signs of solutions must match with data:
  \[ \text{sign}(A\hat{x}) = \text{sign}(Ax) = y \]
  i.e. \( y_i \times (A\hat{x})_i \geq 0 \)

- Penalize mismatches quadratically:
  \[
  \sum_i f (y_i \times (A\hat{x})_i)
  \]
  \[
  f(x) = \begin{cases} 
  x^2 & x \leq 0 \\
  0 & x > 0 
  \end{cases}
  \]

- Also need to normalize after each iteration
  - Makes algorithm non-convex
Matching Sign Pursuit

- Iterative greedy algorithm, similar to CoSaMP
- Maintains running signal estimate and its support $T$.
- MSP Iteration:
  - Identify **sign violations**
  - Compute **proxy**
  - Identify **support**
  - **Consistent Reconstruction** over support

\[
\begin{aligned}
  r &= (\text{diag}(y)A\hat{x})^{-1} \\
  p &= A^Tr \\
  \Omega &= \text{supp}(p|_{2K}) \cup T \\
  b|_{\Omega} &= \arg\min_x \left\| (\text{diag}(y)Ax)^{-1} \right\|_2^2, \\
  \text{s.t. } \|x\|_2 = 1 \text{ and } x|_{T^c} = 0 \\
  \hat{x} &\leftarrow \frac{b|_K}{\|b|_K\|_2}
\end{aligned}
\]
Matching Sign Pursuit (MSP)

(a) Reconstruction Performance

- MSP
- CoSaMP
- Consistent

(b) MSP Reconstruction Improvement

MSP Improvement (dB)

M (bits)
NON-LINEAR DISTORTION
CS with Distorted Measurements

- Distortion is unknown but monotonic
- Monotonicity preserves the order of the measurements
Measurement Ordering

\[ y = f(Ax) \]
Measurement Ordering

\[ y = f(A, x) \]

\[ \tilde{y} = f(\tilde{A}, x) \]

Sorted Measurements
Method 1: MSP on Measurement Difference

\[ \tilde{y} = f \left( \tilde{A} \right) x \]

Sorted Measurements
Method 1: MSP on Measurement Difference

\[ \tilde{y} = f(\tilde{A}) \]

Has positive elements!
Use MSP (1-bit CS) to estimate \( x \).
Alternative Approach: Measurement Substitution

- $A$ is random; $Ax$ is (approximately) normally distributed
- Use order statistics to estimate the measurements
- Method 2:
  - Order measurements
  - Replace measurements with estimate
  - Use standard algorithms

\[
y = f(Ax) = f(A) \cdot x
\]
Measurement Ordering

\[ y = f(A) x \]
Measurement Ordering

\[ y = f(A) \]

Sorted Measurements
Ignore their value!
Measurement Substitution: Order Statistics

- Solve system for $\hat{y} = \tilde{A}x$
- Uses standard CS solvers (convex)
- Can be used in non-CS problems
- Slightly worse performance

Expected value of $i^{th}$ smallest measurement (i.e. MMSE estimate)

$$\hat{y}(i) = \Phi^{-1}(p_i)$$

Uniform spacing $p_i = \frac{i}{M + 1}$

Normal CDF
$y = \tanh(Ax) + \text{noise}$
Wrap Up

• Lessons from Compressive Sensing
  – Non-linear reconstruction allows us to incorporate models
  – Signal model: sparsity; we can extend that further (another talk)
  – Measurement model: Quantization, Saturation, Nonlinearities
  – Randomness helps! Makes everything well behaved.
  – Conventional wisdom might not hold anymore...

• To dig more:
  – Rice Compressive Sensing repository: dsp.rice.edu/cs
  – CS blog: nuit-blanche.blogspot.com
  – E-mail me: petrosb@merl.com