Sparse Signal Reconstruction from Zero Crossings

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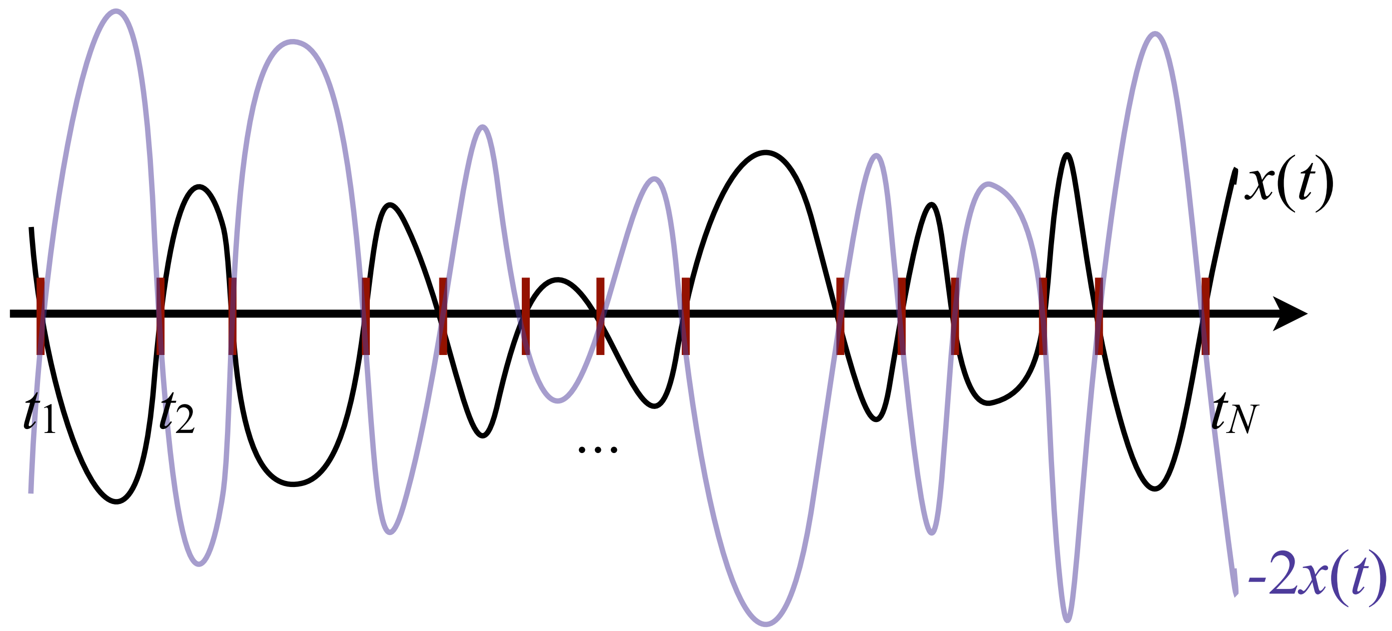
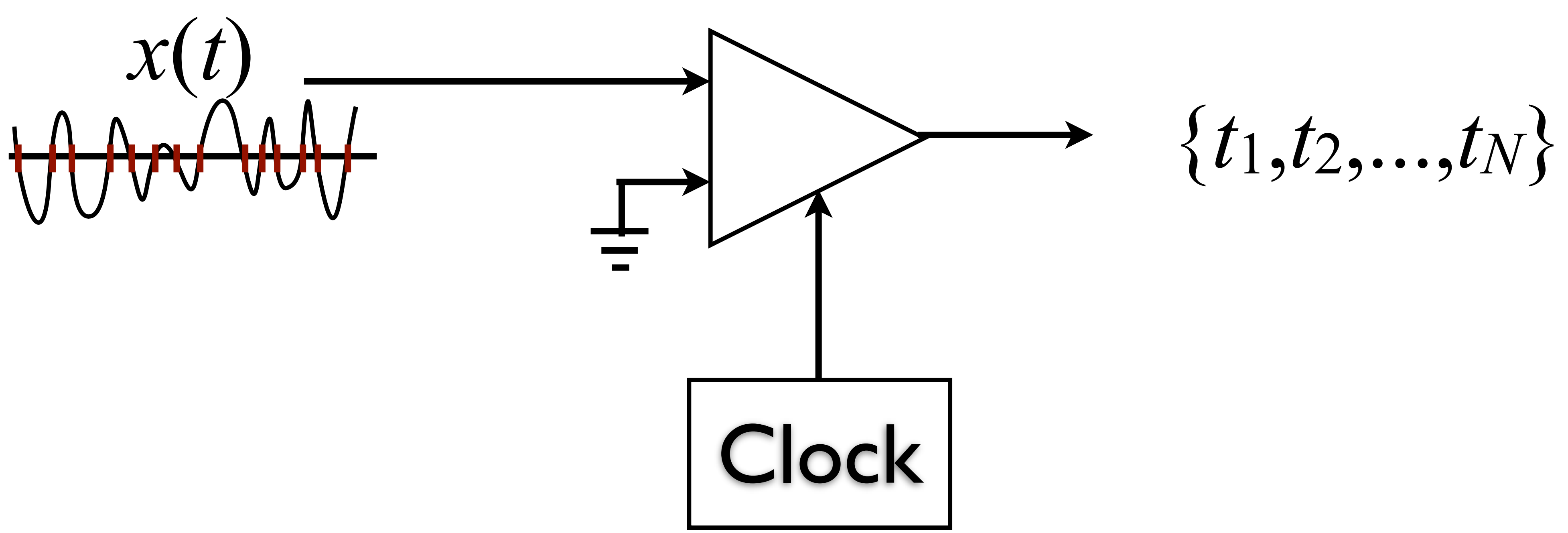
RICE

Signal Reconstruction from Zero Crossings

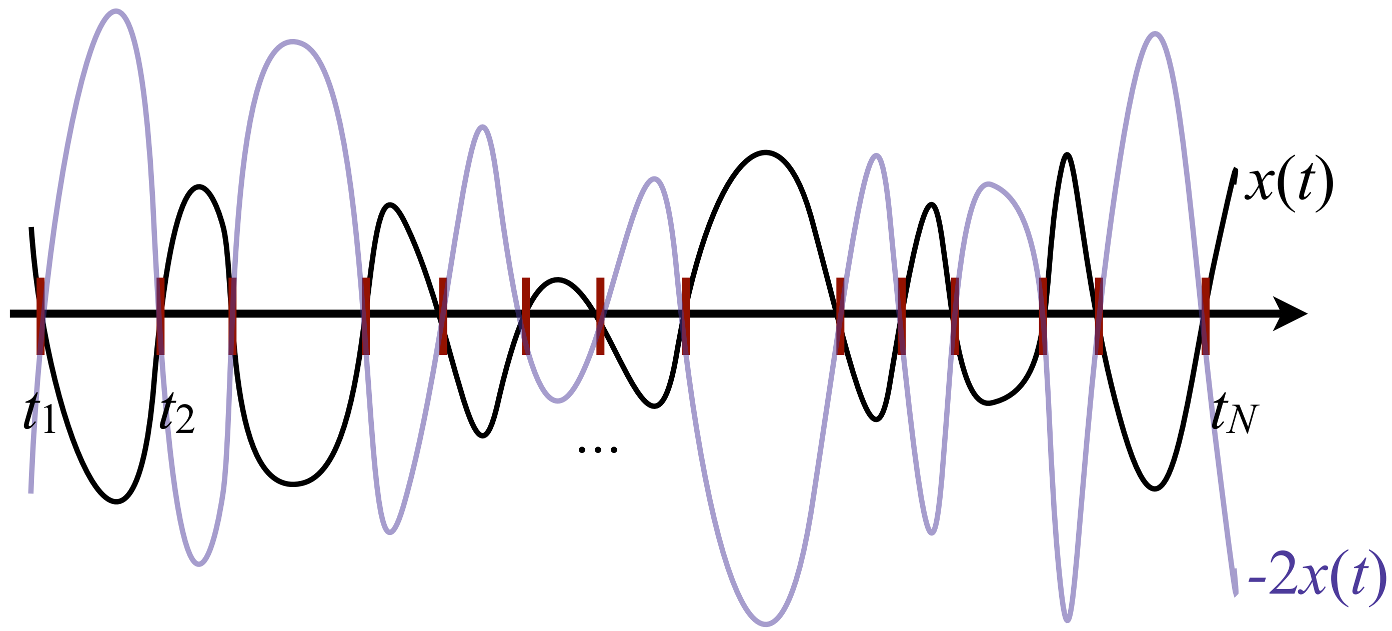


Q: Given only the zero crossings $\{t_1, t_2, \ldots, t_N\}$ of a signal can we reconstruct it?



Easy implementation: only need a comparator and a clock

Logan's Theorem



Q: Given only the zero crossings $\{t_1, t_2, \ldots, t_N\}$ of a signal can we reconstruct it?

Logan's Theorem: YES. Signals bandlimited to $[B, 2B)$ are uniquely determined by their zero crossings.

BUT: an arbitrary set of zero crossings might not correspond to a signal bandlimited to $[B, 2B)$. Reconstruction is not robust. There is ambiguity.

Introduce sparsity to resolve the ambiguity!

Signal Representation

Fourier series of $x(t)$:

$$x(t) = \sum_{n \in \mathcal{B}} \left[a_n \cos(2\pi n t) + b_n \sin(2\pi n t) \right]$$

Vector of coefficients:

$$\mathbf{x} = \begin{bmatrix} a_{n_1} \\ \vdots \\ a_{n_{N/2}} \\ b_{n_1} \\ \vdots \\ b_{n_{N/2}} \end{bmatrix}$$

Sampling Operator

Given $\{t_1, t_2, \ldots, t_N\}$,

$$\mathbf{\Phi}_{\{t_k\}} = \begin{bmatrix} \cos(2\pi n_1 t_1) & \ldots & \cos\left(2\pi n_{N/2} t_1\right) & \sin(2\pi n_1 t_1) & \ldots & \sin\left(2\pi n_{N/2} t_1\right) \\ \cos(2\pi n_1 t_2) & \ldots & \cos\left(2\pi n_{N/2} t_2\right) & \sin(2\pi n_1 t_2) & \ldots & \sin\left(2\pi n_{N/2} t_2\right) \\ \vdots & & \vdots & \vdots & & \vdots \\ \cos(2\pi n_1 t_N) & \ldots & \cos\left(2\pi n_{N/2} t_N\right) & \sin(2\pi n_1 t_N) & \ldots & \sin\left(2\pi n_{N/2} t_N\right) \end{bmatrix}$$

Samples the signal at those times:

$$\mathbf{\Phi}_{\{t_k\}} \mathbf{x} = \begin{bmatrix} x(t_1) \\ \vdots \\ x(t_N) \end{bmatrix}$$

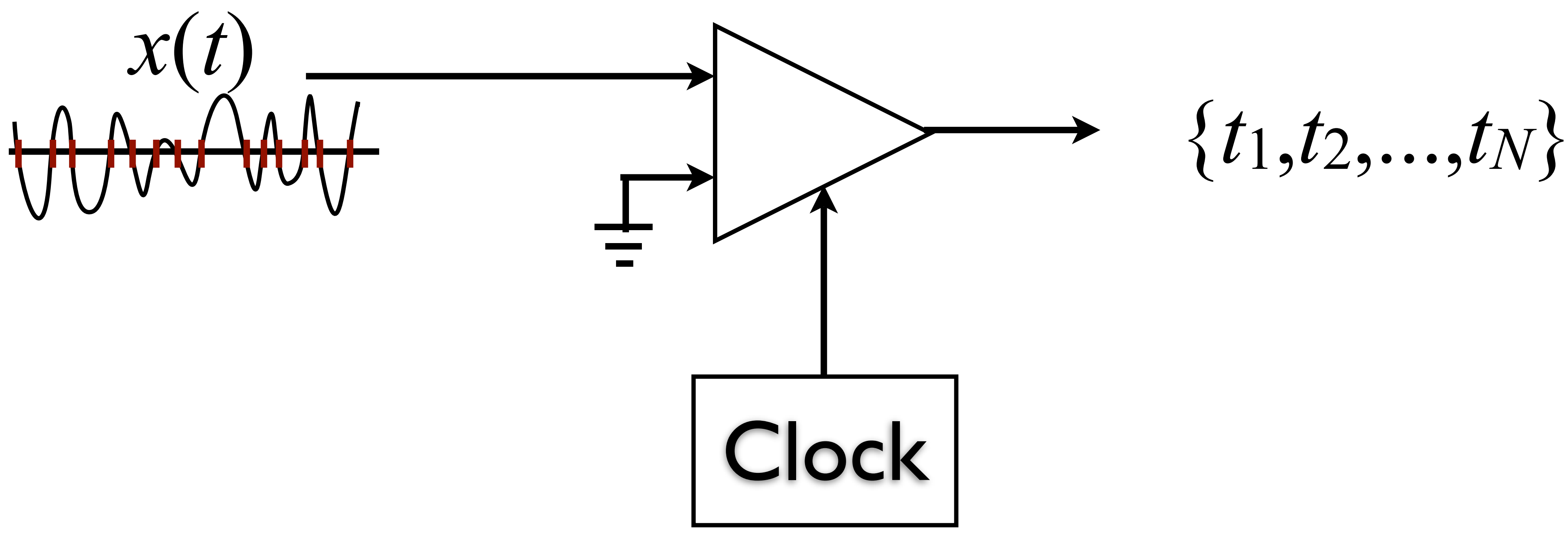
Reconstruction Problem

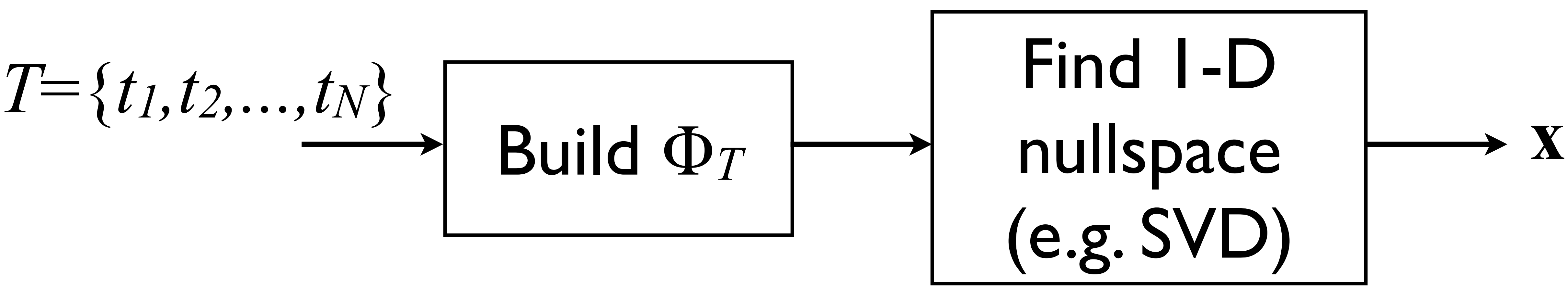
$$\mathbf{\Phi}_{\{t_k\}}\mathbf{x} = \begin{bmatrix} x(t_1) \\ \vdots \\ x(t_N) \end{bmatrix}$$

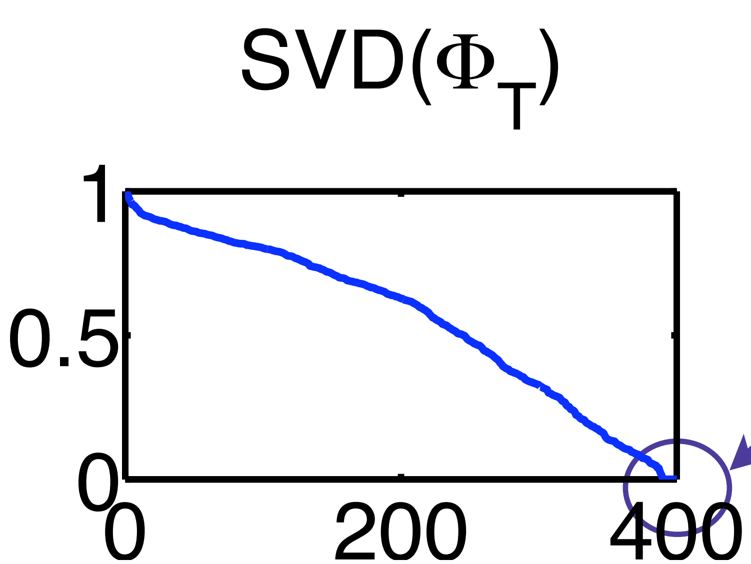
If $T=\{t_1,t_2,\ldots,t_N\}$ are the zero crossings, then the desired signal is in the nullspace of Φ: $\Phi_T\mathbf{x} = 0.$

Logan's theorem$\Rightarrow$ Φ_T has a one-dimensional nullspace.

Signal Acquisition and Reconstruction







In practice: noise and quantization.

No nullspace!

Many small singular values.

Ambiguity!

Sparse Reconstruction

ℓ_1 minimization:

$$\widehat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_1$$

subject to $\Phi\mathbf{x} = 0$

Relaxation:

$$\widehat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_1 + \frac{\lambda}{2}\|\Phi\mathbf{x}\|_2^2$$

Unit energy constraint:

$$\widehat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_1 + \frac{\lambda}{2}\|\Phi\mathbf{x}\|_2^2$$

subject to $\|\mathbf{x}\|_2 = 1$

Fixed Point Equilibrium

$$\widehat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_1 + \frac{\lambda}{2}\|\Phi\mathbf{x}\|_2^2$$

subject to $\|\mathbf{x}\|_2 = 1$

Unconstrained minimization:

$$\text{Cost}(\mathbf{x}) = g(\mathbf{x}) + \frac{\lambda}{2} f(\Phi\mathbf{x})$$

$$\text{Cost}'(\mathbf{x}) = g'(\mathbf{x}) + \frac{\lambda}{2}\Phi^* f'(\Phi\mathbf{x})$$

where:

$$(g'(\mathbf{x}))_i = \begin{cases} -1 & x_i < 0 \\ [-1, 1] & x_i = 0 \\ +1 & x_i > 0 \end{cases}$$

No change if gradients are projected on unit sphere

Minimization Algorithm (based on FPC [Hale, Yin, Zhang, '07])

Big Picture: Gradient descent until equilibrium.

Initialization parameters: $\widehat{\mathbf{x}}, \ \tau$

1. Compute quadratic gradient: $\mathbf{h} = \Phi^T \Phi \widehat{\mathbf{x}}$
2. Project onto sphere: $\mathbf{h}_p = \mathbf{h} - \langle \widehat{\mathbf{x}}, \mathbf{h} \rangle$
3. Quadratic gradient descent: $\widehat{\mathbf{x}} \leftarrow \widehat{\mathbf{x}} - \tau \mathbf{h}_p$
4. Shrink (ℓ_1 gradient descent):

$$\widehat{x}_i \quad \leftarrow \quad \mathrm{sign}(\widehat{x}_i) \max\left\{ |\widehat{x}_i| - \frac{\tau}{\lambda}, 0 \right\}$$

5. Normalize: $\widehat{\mathbf{x}} \quad \leftarrow \quad \dfrac{\widehat{\mathbf{x}}}{\|\widehat{\mathbf{x}}\|}$
6. Iterate until equilibrium.

Optimization on the Sphere

$$\widehat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_1 + \frac{\lambda}{2}\|\Phi\mathbf{x}\|_2^2$$
$$\text{subject to } \|\mathbf{x}\|_2 = 1$$

Optimization is not convex.

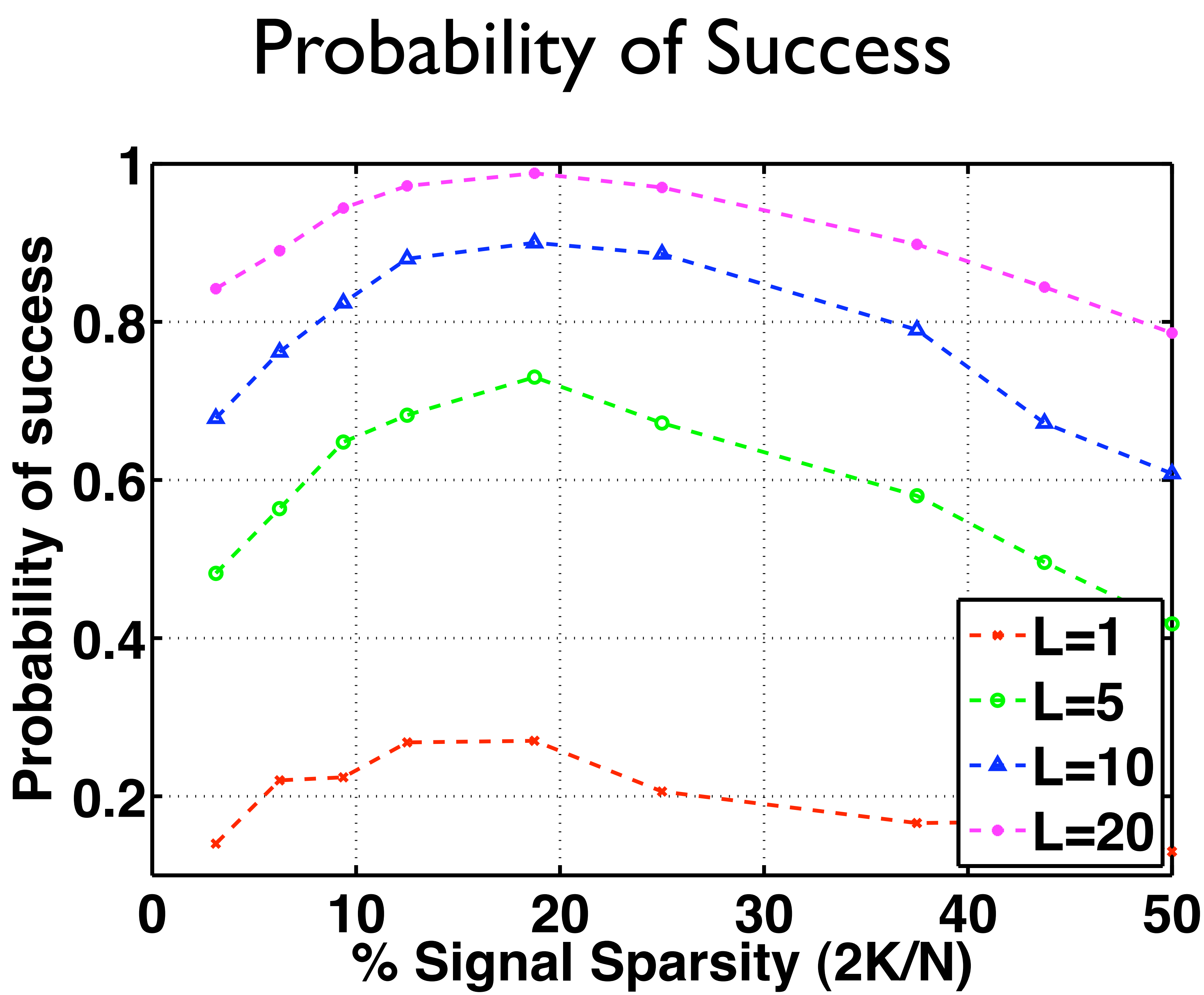
Convergence to global optimum not guaranteed.

Exploit randomness:

- Execute L times with random initializations.
- Pick best solution.

If P=P(success for 1 execution), then
P(overall success)=$1-(1-P)^L$

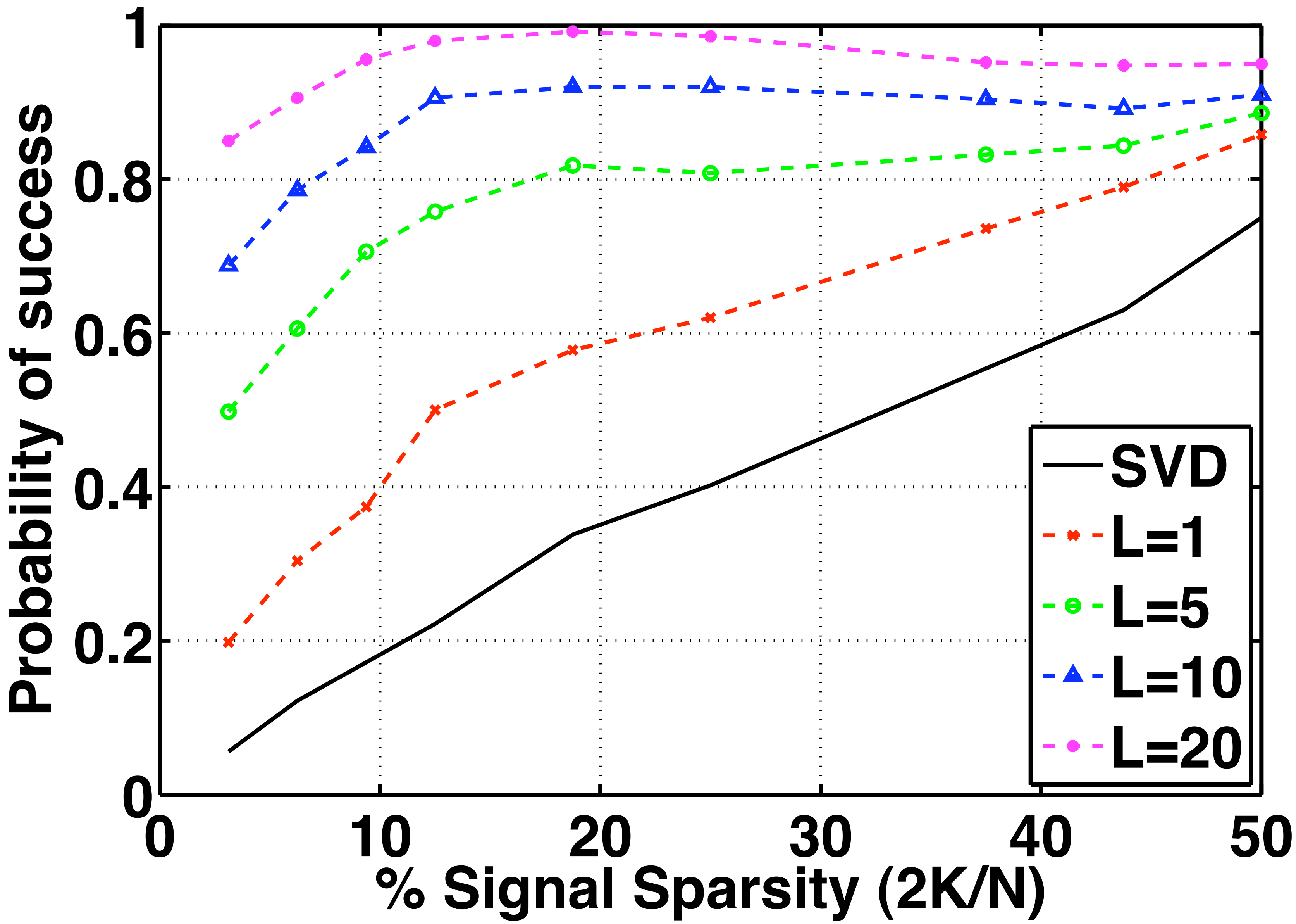
Results



L=number of random initializations
N=256 coefficients

Results

Probability of Success



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Further Relaxation

Optimization on sphere:

$$\widehat{\mathbf{x}} \quad = \quad \arg\min_{\mathbf{x}} \|\mathbf{x}\|_1 + \frac{\lambda}{2}\|\Phi\mathbf{x}\|_2^2$$
$$\text{subject to } \|\mathbf{x}\|_2 = 1$$

Relaxation of sphere constraint:

$$\widehat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_1 + \frac{\lambda_1}{2}\|\Phi\mathbf{x}\|_2^2 + \lambda_2 \left|\|\mathbf{x}\|_2^2 - 1\right|^2$$

We can now use standard ℓ_1 algorithms!

ℓ_1 minimization formulation

$$\widehat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_1 + \frac{\lambda_1}{2}\|\Phi\mathbf{x}\|_2^2 + \lambda_2 \left|\|\mathbf{x}\|_2^2 - 1\right|^2$$

Let:

$$\widetilde{\Phi} = \left[\begin{array}{c} c\mathbf{x} \\ \Phi \end{array}\right], \quad c = \left(\frac{\lambda_2}{\lambda_1}\right)^2$$

At equilibrium:

$$\widehat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_1 + \frac{\lambda_1}{2}\left\|\widetilde{\Phi}\mathbf{x} - \left[\begin{array}{c} c \\ \mathbf{0} \end{array}\right]\right\|_2^2$$

Reweighted FPC algorithm (with Chinmay Hegde)

Initialization parameters: $\widehat{\mathbf{x}},\ \lambda_1,\ \lambda_2$

1. Build $\widetilde{\Phi} = \begin{bmatrix} c\widehat{\mathbf{x}} \\ \Phi \end{bmatrix},\ c = \left(\frac{\lambda_2}{\lambda_1}\right)^2$

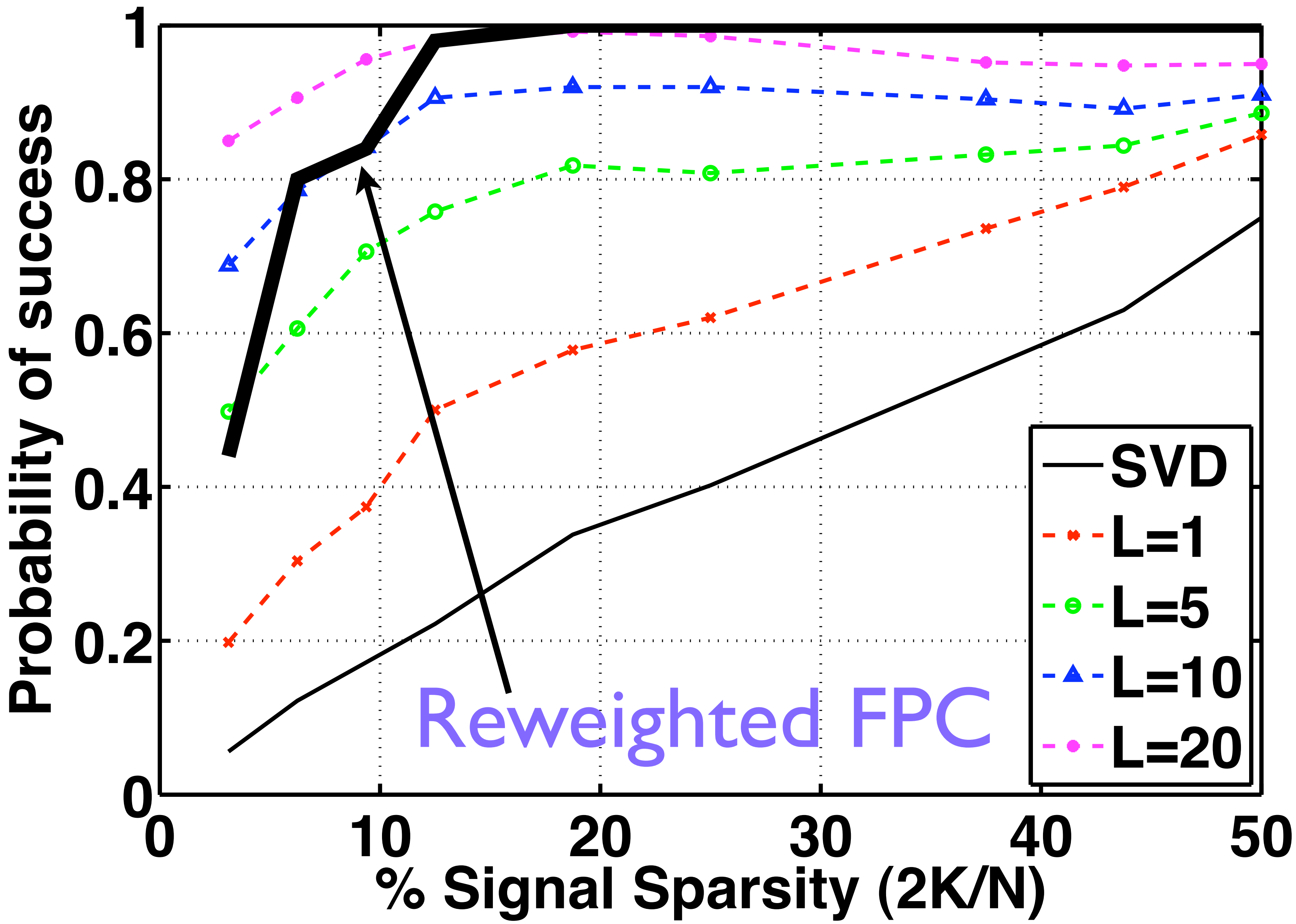
2. Estimate using FPC:

$$\widehat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_1 + \frac{\lambda_1}{2}\left\|\widetilde{\Phi}\mathbf{x} - \begin{bmatrix} c \\ \mathbf{0} \end{bmatrix}\right\|_2^2$$

3. Iterate until equilibrium.

Results

Probability of Success



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Conclusions

- Signal reconstruction from zero crossings is unstable.
- Sparsity assumptions can be used to stabilize it.
- Reconstruction requires optimization on the sphere
- The problem is non-convex. Convergence to global optimum is not guaranteed.
- We provide two algorithms that converge experimentally with very high probability.