

# Robust 1-Bit Compressive Sensing via Binary Stable Embeddings of Sparse Vectors

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**Abstract**—The Compressive Sensing (CS) framework aims to ease the burden on analog-to-digital converters (ADCs) by reducing the sampling rate required to acquire and stably recover sparse signals. Practical ADCs not only sample but also quantize each measurement to a finite number of bits; moreover, there is an inverse relationship between the achievable sampling rate and the bit-depth. In this paper, we investigate an alternative CS approach that shifts the emphasis from the sampling rate to the number of bits per measurement. In particular, we explore the extreme case of 1-bit CS measurements, which capture just their sign. Our results come in two flavors. First, we consider ideal reconstruction from noiseless 1-bit measurements and provide a lower bound on the best achievable reconstruction error. We also demonstrate that i.i.d. random Gaussian matrices provide measurement mappings that, with overwhelming probability, achieve nearly optimal error decay. Next, we consider reconstruction robustness to measurement errors and noise and introduce the *Binary  $\epsilon$ -Stable Embedding* (BeSE) property, which characterizes the robustness of the measurement process to sign changes. We show that the same class of matrices that provide almost optimal noiseless performance also enable such a robust mapping. On the practical side, we introduce the *Binary Iterative Hard Thresholding* (BIHT) algorithm for signal reconstruction from 1-bit measurements that offers state-of-the-art performance.

## I. INTRODUCTION

Recent advances in signal acquisition theory have led to significant interest in alternative sampling methods. Specifically, conventional sampling systems rely on the Shannon sampling theorem that states that signals must be sampled uniformly at the Nyquist rate, i.e., a rate twice their bandwidth. However, the *compressive sensing*

(CS) framework describes how to reconstruct a signal  $\mathbf{x} \in \mathbb{R}^N$  from the linear measurements

$$\mathbf{y} = \Phi \mathbf{x}, \quad (1)$$

where  $\Phi \in \mathbb{R}^{M \times N}$  with  $M < N$  is an underdetermined measurement system [1, 2]. It is possible to design a physical sampling system  $\Phi$  such that  $\mathbf{y} = \Phi \mathbf{x} = \bar{\Phi}(x(t))$  where  $\mathbf{x}$  is a vector of Nyquist-rate samples of a bandlimited signal  $x(t)$ ,  $t \in \mathbb{R}$ . In this case, (1) translates to low, sub-Nyquist sampling rates, providing the framework’s axial significance: CS enables the acquisition and accurate reconstruction of signals that were previously out of reach, limited by hardware sampling rates [3] or number of sensors [4].

Although inversion of (1) seems ill-posed, it has been demonstrated that  $K$ -sparse signals, i.e.,  $\mathbf{x} \in \Sigma_K$  where  $\Sigma_K := \{\mathbf{x} \in \mathbb{R}^N : \|\mathbf{x}\|_0 := |\text{supp}(\mathbf{x})| \leq K\}$ , can be reconstructed exactly [1, 2]. To do this, we could naively solve for the sparsest signal that satisfies (1),

$$\mathbf{x}^* = \underset{\mathbf{u} \in \mathbb{R}^N}{\text{argmin}} \|\mathbf{u}\|_0 \text{ s.t. } \mathbf{y} = \Phi \mathbf{u}; \quad (\text{R}_{\text{CS}})$$

however, this non-convex program exhibits combinatorial complexity in the size of the problem [5]. Instead, we solve *Basis Pursuit* (BP) by relaxing the objective in (R<sub>CS</sub>) to the  $\ell_1$ -norm; the result is a convex, polynomial-time algorithm [6]. A key realization is that, under certain conditions on  $\Phi$ , the BP solution will be equivalent to that of (R<sub>CS</sub>) [1]. This basic reconstruction framework has been expanded to include numerous fast algorithms as well as provably robust algorithms for reconstruction from noisy measurements [7–11]. Reconstruction can also be performed with iterative and greedy methods [12–14].

Reconstruction guarantees for BP and other algorithms are often demonstrated for  $\Phi$  that are endowed with the *restricted isometry property* (RIP), the sufficient condition that the norm of the measurements is close to the norm of the signal for all sparse  $\mathbf{x}$  [7].<sup>1</sup> This can

<sup>1</sup>The RIP is in fact not needed to demonstrate exact reconstruction guarantees in noiseless settings, however it proves quite useful for establishing robust reconstruction guarantees in noise.

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be expressed, in general terms, as a  $\delta$ -stable embedding. Let  $\delta \in (0, 1)$  and  $\mathcal{X}, \mathcal{S} \subset \mathbb{R}^N$ . We say the mapping  $\Phi$  is a  $\delta$ -stable embedding of  $\mathcal{X}, \mathcal{S}$  if

$$(1 - \delta) \|\mathbf{x} - \mathbf{s}\|_2^2 \leq \|\Phi \mathbf{x} - \Phi \mathbf{s}\|_2^2 \leq (1 + \delta) \|\mathbf{x} - \mathbf{s}\|_2^2, \quad (2)$$

for all  $\mathbf{x} \in \mathcal{X}$  and  $\mathbf{s} \in \mathcal{S}$ . The RIP requires that (2) hold for all  $\mathbf{x}, \mathbf{s} \in \Sigma_K$ ; that is, it is a stable embedding of sparse vectors. A key result in the CS literature is that, if the coefficients of  $\Phi$  are randomly drawn from a sub-Gaussian distribution, then  $\Phi$  will satisfy the RIP with high probability as long as  $M \geq C_\delta K \log(N/K)$ , for some constant  $C_\delta$  [15]. Several hardware inspired designs with only a few randomized components have also been shown to satisfy this property [3, 17–19].

In practice, CS measurements must be quantized, i.e., each measurement is mapped from a real value (over a potentially infinite range) to a discrete value over some finite range. For example, in uniform quantization, a measurement is mapped to one of  $2^B$  distinct values, where  $B$  denotes the number of bits per measurement. Quantization is an irreversible process that introduces error in the measurements. One way to account for quantization error is to treat it as bounded noise and employ robust reconstruction algorithms. Alternatively, we might try to reduce the error by choosing the most efficient quantizer for the distribution of the measurements. Several reconstruction techniques that specifically address CS quantization have also been proposed [20–24].

While quantization error is a minor inconvenience, fine quantization invokes a more burdensome, yet often overlooked source of adversity: in hardware systems, it is the primary bottleneck limiting sample rates [25, 26]. In other words, the analog-to-digital converter (ADC) is beholden to the quantizer. First, quantization significantly limits the maximum speed of the ADC, forcing an exponential decrease in sampling rate as the number of bits is increased linearly [26]. Second, the quantizer is the primary power consumer in an ADC. Thus, more bits per measurement directly translates to slower sampling rates and increased ADC costs. Third, fine quantization is more susceptible to non-linear distortion in the ADC electronics, requiring explicit treatment in the reconstruction [27]. As we have seen, the CS framework provides one mechanism to alleviate the quantization bottleneck by reducing the ADC sampling rate. Is it possible to extend the CS framework to mitigate this problem directly in the quantization domain by reducing the number of bits per measurement (bit-depth) instead?

In this paper we concretely answer this question in the affirmative. We consider an extreme quantization; just

one bit per CS measurement, representing its sign. The quantizer is thus reduced to a simple comparator that tests for values above or below zero, enabling extremely simple, efficient, and fast quantization. A 1-bit quantizer is also more robust to a number of commonly encountered non-linear distortions in the input electronics, as long as they preserve the signs of the measurements.

It is not obvious that the signs of the CS measurements retain enough information for signal reconstruction; for example, it is immediately clear that the scale (absolute amplitude) of the signal is lost. Nonetheless, there is strong empirical evidence that signal reconstruction is possible [27–30]. In this paper we develop strong theoretical reconstruction and robustness guarantees for a decoder able to enforce both sparsity and consistency of the reconstructed vector, in the same spirit as classical guarantees provided in CS by the RIP.

We briefly describe the 1-bit CS framework proposed in [28]. Measurements of a signal  $\mathbf{x} \in \mathbb{R}^N$  are computed via

$$\bar{\mathbf{y}} = A(\mathbf{x}) := \text{sign}(\Phi \mathbf{x}), \quad (3)$$

where the sign operator is applied component wise on  $\Phi \mathbf{x}$ , where  $\text{sign} \lambda$  equals 1 if  $\lambda > 0$  and  $-1$  otherwise, for any  $\lambda \in \mathbb{R}$ . Thus, the measurement operator  $A(\cdot)$  is a mapping from  $\mathbb{R}^N$  to the Boolean cube<sup>2</sup>  $\mathcal{B}^M := \{-1, 1\}^M$ . At best, we hope to recover signals  $\mathbf{x} \in \Sigma_K^* := \{\mathbf{x} \in S^{N-1} : \|\mathbf{x}\|_0 \leq K\}$  where  $S^{N-1} := \{\mathbf{x} \in \mathbb{R}^N : \|\mathbf{x}\|_2 = 1\}$  is the unit hypersphere of dimension  $N$ . We restrict our attention to sparse signals on the unit sphere since, as previously mentioned, the scale of the signal has been lost during the quantization process. To reconstruct, we enforce *consistency* on the signs of the estimate's measurements, i.e., that  $A(\mathbf{x}^*) = A(\mathbf{x})$ . Specifically, we define a general non-linear reconstruction decoder  $\Delta^{1\text{bit}}(\bar{\mathbf{y}}, \Phi, K)$  such that, for  $\mathbf{x}^* = \Delta^{1\text{bit}}(\bar{\mathbf{y}}, \Phi, K)$ , the solution  $\mathbf{x}^*$  is

- (i) sparse, i.e., satisfies  $\|\mathbf{x}^*\|_0 \leq K = \|\mathbf{x}\|_0$ ,
- (ii) consistent, i.e., satisfies  $A(\mathbf{x}^*) = \bar{\mathbf{y}} = A(\mathbf{x})$ .

With  $(\mathbf{R}_{\text{CS}})$  from CS as a guide, one candidate program for reconstruction that respects these two conditions is

$$\mathbf{x}^* = \underset{\mathbf{u} \in S^{N-1}}{\text{argmin}} \|\mathbf{u}\|_0 \text{ s.t. } \bar{\mathbf{y}} = \text{sign}(\Phi \mathbf{u}). \quad (\mathbf{R}_{1\text{BCS}})$$

Although the parameter  $K$  is not explicit in  $(\mathbf{R}_{1\text{BCS}})$ , the solution will be  $K'$ -sparse with  $K' \leq K$  because  $\mathbf{x}$  is a feasible point of the constraints.

<sup>2</sup>Generally, the  $M$ -dimensional Boolean cube is defined as  $\{0, 1\}^M$ . Without loss of generality, we use  $\{-1, 1\}^M$  instead.

Since  $(R_{1\text{BCS}})$  is computationally intractable, [28] proposes a relaxation that replaces the objective with the  $\ell_1$ -norm and enforces consistency via a linear convex constraint. However, the resulting program remains non-convex due to the unit-sphere requirement. Be that as it may, several optimization algorithms have been developed for the relaxation, as well as a greedy algorithm inspired by the same ideas [28–30]. While previous empirical results from these algorithms provide motivation for the validity of this 1-bit framework, there have been few analytical guarantees to date.

The primary contribution of this paper is a rigorous analysis of the 1-bit CS framework. Specifically, we examine how the reconstruction error behaves as we increase the number of measurement bits  $M$  given the signal dimension  $N$  and sparsity  $K$ . We provide two flavors of results. First, we determine a lower bound on reconstruction performance from all possible mappings  $A$  with any reconstruction decoder, i.e., the best achievable performance of this 1-bit CS framework. We further demonstrate that if the elements of  $\Phi$  are drawn randomly from Gaussian distribution or its rows are drawn uniformly from the unit sphere, then the worst-case reconstruction error using  $\Delta^{1\text{bit}}$  will decay at a rate almost optimal with the number of measurements, up to a log factor in the oversampling rate  $M/K$  and the signal dimension  $N$ . Second, we provide conditions on  $A$  that enable us to characterize the reconstruction performance even when some of the measurement signs have changed (e.g., due to noise in the measurements). In other words, we derive the conditions under which robust reconstruction from 1-bit measurements can be achieved. We do so by demonstrating that  $A$  is a stable embedding of sparse signals, similar to the RIP. We apply these stable embedding results to the cases where we have noisy measurements and signals that are not strictly sparse. Our guarantees demonstrate that the 1-bit CS framework is on sound footing and provide a first step toward analysis of the relaxed 1-bit techniques used in practice.

To develop robust reconstruction guarantees, we propose a new tool, the *binary  $\epsilon$ -stable embedding* (B $\epsilon$ SE), to characterize 1-bit CS systems. The B $\epsilon$ SE implies that the normalized angle between any sparse vectors in  $S^{N-1}$  is close to the normalized Hamming distance between their 1-bit measurements. We demonstrate that the same class of random  $A$  as above exhibit this property when  $M \geq C_\epsilon K \log N$  (where  $C_\epsilon$  is some constant). Thus remarkably, there exist  $A$  such that the B $\epsilon$ SE holds when both the number of measurements  $M$  is smaller than the dimension of the signal  $N$  and the

measurement bit-depth is at minimum.

As a complement to our theoretical analysis, we introduce a new 1-bit CS reconstruction algorithm, *Binary Iterative Hard Thresholding* (BIHT). Via simulations, we demonstrate that BIHT yields a significant improvement in both reconstruction error as well as consistency, as compared with previous algorithms. To gain intuition about the behavior of BIHT, we explore the way that this algorithm enforces consistency and compare and contrast it with previous approaches. Perhaps more important than the algorithm itself is the discovery that the BIHT consistency formulation provides a significantly better feasible solution in noiseless settings, as compared with previous algorithms. Finally, we provide a brief explanation regarding why this new formulation achieves better solutions, and its connection with results in the machine learning literature.

Since the first appearance of this work, Plan and Vershynin have developed additional theoretical results and bounds on the performance of 1-bit CS, as well as two convex algorithms with theoretical guarantees [31–33]. The results in [31, 32] generalize the B $\epsilon$ SE guarantees for more general classes of signals, including compressible signals in addition to simply sparse ones. However, the guarantees provided in that work exhibit worse decay rates in the error performance and the tightness of the B $\epsilon$ SE property. Furthermore, the results of [32, 33] are intimately tied to reconstruction algorithms, in contrast to our analysis. We point out similarities and differences with our results when appropriate in the subsequent development.

In addition to benchmarking the performance of BIHT, our simulations demonstrate that many of the theoretical predictions that arise from our analysis (such as the error rate as a function of the number of measurements or the error rate as a function of measurement Hamming distance), are actually exhibited in practice. This suggests that our theoretical analysis is accurately explaining the true behavior of the framework.

The remainder of this paper is organized as follows. In Section II, we develop performance results for 1-bit CS in the noiseless setting. Specifically we develop a lower bound on reconstruction performance as well as provide the guarantee that Gaussian matrices enable this performance. In Section III we introduce the notion of a B $\epsilon$ SE for the mapping  $A$  and demonstrate that Gaussian matrices facilitate this property. We also expand reconstruction guarantees for measurements with Gaussian noise (prior to quantization) and non-sparse signals. To make use of these results in practice, in Section IV we

present the BIHT algorithm for practical 1-bit reconstruction. In Section V we provide simulations of BIHT to verify our claims. In Section VI we conclude with a discussion about implications and future extensions. To facilitate the flow of the paper and clear descriptions of the results, most of our proofs are provided in the appendices.

## II. NOISELESS RECONSTRUCTION PERFORMANCE

### A. Reconstruction performance lower bounds

In this section, we seek to provide guarantees on the reconstruction error from 1-bit CS measurements. Before analyzing this performance from a specific mapping  $A$  with the consistent sparse reconstruction decoder  $\Delta^{1\text{bit}}(\bar{\mathbf{y}}, \Phi, K)$ , it is instructive to determine the best achievable performance from measurements acquired using any mapping. Thus, in this section we seek a lower bound on the reconstruction error.

We develop the lower bound on the reconstruction error based on how well the quantizer exploits the available measurement bits. A distinction we make in this section is that of measurement bits, which is the number of bits acquired by the measurement system, versus information bits, which represent the true amount of information carried in the measurement bits. Our analysis follows similar ideas to that in [34, 35], adapted to sign measurements.

We first examine how 1-bit quantization operates on the measurements. Specifically, we consider the orthants of the measurement space. An orthant in  $\mathbb{R}^M$  is the set of vectors such that all the vector's coefficients have the same sign pattern

$$\mathcal{O}_{\bar{z}} := \{z \in \mathbb{R}^M \mid \text{sign } z = \bar{z}\}, \quad (4)$$

where  $\bar{z} \in \mathcal{B}^M$ . Notice that  $\cup_{\bar{z} \in \mathcal{B}^M} \mathcal{O}_{\bar{z}} = \mathbb{R}^M$  and  $\mathcal{O}_{\bar{z}} \cap \mathcal{O}_{\bar{z}'} = \emptyset$  if  $\bar{z} \neq \bar{z}'$ . Therefore, any  $M$ -dimensional space is partitioned to  $2^M$  orthants. Figure 1(a) shows the 8 orthants of  $\mathbb{R}^3$  as an example. Since 1-bit quantization only preserves the signs of the measurements, it encodes in which measurement space orthant the measurements lie. Thus, each available quantization point corresponds to an orthant in the measurement space. Any unquantized measurement vector  $\Phi x$  that lies in an orthant of the measurement space will quantize to the corresponding quantization point of that orthant and cannot be distinguished from any other measurement vector in the same orthant. To obtain a lower bound on the reconstruction error, we begin by bounding the number of quantization points (or equivalently the number of orthants) that are used to encode the signal.

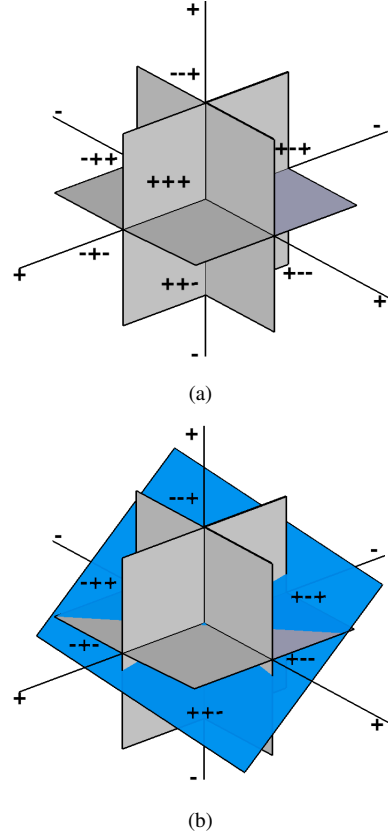


Fig. 1: (a) The 8 orthants in  $\mathbb{R}^3$ . (b) Intersection of orthants by a 2-dimensional subspace. At most 6 of the 8 available orthants are intersected.

While there are generally  $2^M$  orthants in the measurement space, the space formed by measuring all sparse signals occupies a small subset of the available orthants. We determine the number of available orthants that can be intersected by the measurements in the following lemma:

**Lemma 1.** *Let  $x \in \mathcal{S} := \cup_{i=1}^L \mathcal{S}_i$  belong to a union of  $L$  subspaces  $\mathcal{S}_i \subset \mathbb{R}^N$  of dimension  $K$ , and let  $M \geq 2K$  1-bit measurements  $\bar{\mathbf{y}}$  be acquired via the mapping  $A : \mathbb{R}^N \rightarrow \mathcal{B}^M$  as defined in (3). Then the measurements  $\bar{\mathbf{y}}$  can effectively use at most  $2^K L \binom{M}{K}$  quantization points, i.e., carry at most  $K \log_2(2Me/K) + \log_2(L)$  information bits.*

*Proof:* A  $K$ -dimensional subspace in an  $M$ -dimensional space cannot lie in all the  $2^M$  available octants. For example, as shown in Fig. 1(b), a 2-dimensional subspace of a 3-dimensional space can intersect at most 6 of the available octants. In Appendix A, we demonstrate that one arbitrary  $K$ -dimensional subspace

in an  $M$ -dimensional space intersects at most  $2^K \binom{M}{K}$  orthants of the  $2^M$  available. Since  $\Phi$  is a linear operator, any  $K$ -dimensional subspace  $\mathcal{S}_i$  in the signal space  $\mathbb{R}^N$  is mapped through  $\Phi$  to a subspace  $\mathcal{S}'_i = \Phi \mathcal{S}_i \subset \mathbb{R}^M$  that is also at most  $K$ -dimensional and therefore follows the same bound. Thus, if the signal of interest belongs in a union  $\mathcal{S} := \bigcup_{i=1}^L \mathcal{S}_i$  of  $L$  such  $K$ -dimensional subspaces, then  $\Phi \mathbf{x} \in \mathcal{S}' := \bigcup_{i=1}^L \mathcal{S}'_i$ , and it follows that at most  $2^K L \binom{M}{K}$  orthants are intersected. This means that at most  $2^K L \binom{M}{K} \leq L \left(\frac{2eM}{K}\right)^K$  effective quantization points can be used, i.e., at most  $K \log_2(2eM/K) + \log_2(L)$  information bits can be obtained. ■

Since  $K$ -sparse signals in any basis  $\Psi \in \mathbb{R}^{N \times N}$  belong to a union of at most  $\binom{N}{K}$  subspaces in  $\mathbb{R}^N$  with  $\binom{N}{K} \leq (eN/K)^K$ , using Lemma 1 we can obtain the following corollary.<sup>3</sup>

**Corollary 1.** *Let  $\mathbf{x} = \Psi \boldsymbol{\alpha} \in \mathbb{R}^N$  be  $K$ -sparse in a certain basis  $\Psi \in \mathbb{R}^{N \times N}$ , i.e.,  $\boldsymbol{\alpha} \in \Sigma_K$ . Then the measurements  $\bar{\mathbf{y}} = A(\mathbf{x})$  can effectively use at most  $2^K \binom{N}{K} \binom{M}{K}$  1-bit quantization points, i.e., carry at most  $2K \log_2(\sqrt{2} e \sqrt{NM}/K)$  information bits.*

The set of signals of interest to be encoded is the set of unit-norm  $K$ -sparse signals  $\Sigma_K^*$ . Since unit-norm signals of a  $K$ -dimensional subspace form a  $K$ -dimensional unit sphere in that subspace,  $\Sigma_K^*$  is a union of  $\binom{N}{K}$  such unit spheres. Then, at most  $Q := 2^K \binom{N}{K} \binom{M}{K}$  available quantization points partition  $\Sigma_K^*$  into at most  $Q$  smaller sets, each of which contains all the signals that quantize to the same point.

To develop the lower bound on the reconstruction error we examine how  $\Sigma_K^*$  can be optimally partitioned with respect to the worst-case error, given the number of quantization points used. The measurement and reconstruction process maps each signal in  $\Sigma_K^*$  to a finite set of quantized signals  $\mathcal{Q} \subset \Sigma_K^*$ ,  $|\mathcal{Q}| = Q$ . At best this map ensures that the worst case reconstruction error is minimized, i.e.,

$$\epsilon_{\text{opt}} = \max_{\mathbf{x} \in \Sigma_K^*} \min_{\mathbf{q} \in \mathcal{Q}} \|\mathbf{x} - \mathbf{q}\|_2, \quad (5)$$

where  $\epsilon_{\text{opt}}$  denotes the worst-case quantization error and  $\mathbf{q}$  each of the available quantization points. The optimal lower bound is achieved by designing  $\mathcal{Q}$  to minimize (5) without considering whether the measurement and reconstruction process actually achieve this design. Thus, designing the set  $\mathcal{Q}$  becomes a set covering problem. Appendix B makes this intuition precise and proves the following statement.

<sup>3</sup>This corollary is easily adaptable to a redundant frame  $\Psi \in \mathbb{R}^{N \times D}$  with  $D \geq N$ .

**Theorem 1.** *Let the mapping  $A : \mathbb{R}^N \rightarrow \mathcal{B}^M$  and let  $\mathbf{x} \in \Sigma_K^*$ . Then any reconstruction decoder estimating  $\mathbf{x}$  from  $\bar{\mathbf{y}} = A(\mathbf{x})$  has error defined by (5) of at least*

$$\epsilon_{\text{opt}} \geq \frac{K}{2eM + \sqrt{2}K^{3/2}} \xrightarrow{M} \Omega\left(\frac{K}{M}\right).$$

Thus, when  $M$  is high compared to  $K^{3/2}$ , the worst-case error cannot decay at a rate faster than  $\Omega(1/M)$  as a function of the number measurements, no matter what reconstruction algorithm is used.

This result assumes noiseless acquisition and provides no guarantees of robustness and noise resiliency. This is in line with existing results on scalar quantization in oversampled representations and CS that state that the distortion due to scalar quantization of noiseless measurements cannot decrease faster than the inverse of the measurement rate [34–38]. Notice the absence of the ambient space dimension  $N$  in the lower bound of  $\epsilon_{\text{opt}}$ . It is not clear whether that absence is an artifact of the proof or can be improved with a tighter bound.

To improve the rate vs. distortion trade-off, alternative quantization methods must be used, such as Sigma-Delta ( $\Sigma\Delta$ ) quantization [39–45] or non-monotonic scalar quantization [46]. Specifically,  $\Sigma\Delta$  approaches to CS can achieve error decay rate of  $O((K/M)^{p-1/2})$ , where  $p$  is the order of the quantizer [45]. However,  $\Sigma\Delta$  quantization requires feedback during the quantization process, which is not necessary in scalar quantization. Furthermore, the result in [45] only holds for multibit quantizers, not 1-bit ones. While efficient 1-bit  $\Sigma\Delta$  quantization has been shown for classical sampling [40, 47], to the best of our knowledge, similar results are not currently known for 1-bit  $\Sigma\Delta$  in CS applications. Alternatively, non-monotonic scalar quantization can achieve error decay exponential in the number of measurements  $M$ , even in CS applications [46]. However, such a scheme requires a significantly more complex scalar quantizer and reconstruction approach [48].

Theorem 1 bounds the best possible performance of any decoder over all possible mappings  $A$  applied to vectors in  $\Sigma_K^*$ , and so it bounds the one of a  $\Delta^{1\text{bit}}$  decoder. However, not all mappings  $A$  will behave as the lower bound suggests. In the next section we identify two classes of matrices such that the mapping  $A$  admits an upper bound on the reconstruction error from a general decoder  $\Delta^{1\text{bit}}$  that decays almost optimally.

### B. Achievable performance via random projections

In this section we describe a class of matrices  $\Phi$  such that the consistent sparse reconstruction decoder

$\Delta^{\text{1bit}}(\bar{\mathbf{y}}, \Phi, K)$  can indeed achieve error decay rates of optimal order, described by Theorem 1, with the number of measurements growing linearly in the sparsity  $K$  and logarithmically in the dimension  $N$ , as is required in conventional CS. We first focus our analysis on Gaussian matrices, i.e.,  $\Phi$  such that each element  $\phi_{i,j}$  is randomly drawn i.i.d. from the standard Gaussian distribution,  $\mathcal{N}(0, 1)$ . In the rest of the paper, we use the short notation  $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$  to characterize such matrices, and we write  $\varphi \sim \mathcal{N}^{N \times 1}(0, 1)$  to describe the equivalent random vectors in  $\mathbb{R}^N$  (e.g., the rows of  $\Phi$ ). For these matrices  $\Phi$ , we prove the following in Appendix C.

**Theorem 2.** *Let  $\Phi$  be matrix generated as  $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$ , and let the mapping  $A : \mathbb{R}^N \rightarrow \mathcal{B}^M$  be defined as in (3). Fix  $0 \leq \eta \leq 1$  and  $\epsilon_o > 0$ . If the number of measurements is*

$$M \geq \frac{2}{\epsilon_o} (2K \log(N) + 4K \log(\frac{17}{\epsilon_o}) + \log \frac{1}{\eta}), \quad (6)$$

then for all  $\mathbf{x}, \mathbf{s} \in \Sigma_K^*$  we have that

$$\|\mathbf{x} - \mathbf{s}\|_2 > \epsilon_o \Rightarrow A(\mathbf{x}) \neq A(\mathbf{s}), \quad (7)$$

or equivalently

$$A(\mathbf{x}) = A(\mathbf{s}) \Rightarrow \|\mathbf{x} - \mathbf{s}\|_2 \leq \epsilon_o,$$

with probability higher than  $1 - \eta$ .

Theorem 2 is a uniform reconstruction result, meaning that with high probability all vectors  $\mathbf{x}, \mathbf{s} \in \Sigma_K^*$  can be reconstructed as opposed to a non-uniform result where each vector could be reconstructed with high probability.

As derived in Appendix G, Theorem 2 demonstrates that if we use Gaussian matrices in the mapping  $A$ , then, given a fixed probability level  $\eta$ , the reconstruction decoder  $\Delta^{\text{1bit}}(\bar{\mathbf{y}}, \Phi, K)$  will recover signals with error order

$$\epsilon_o = O\left(\frac{K}{M} \log \frac{MN}{K}\right),$$

which decays almost optimally compared to the lower bound given in Theorem 1 up to a log factor in  $MN/K$ . Whether the gap can be closed, with tighter lower or upper bounds is still an open question. Notice that the hidden proportionality factor in this last relation depends linearly on  $\log 1/\eta$  which is assumed fixed.

We should also note a few minor extensions of Theorem 2. We can multiply the rows of  $\Phi$  with a positive scalar without changing the signs of the measurements. By normalizing the rows of the Gaussian matrix we obtain another class of matrices, ones with rows drawn uniformly from the unit  $\ell_2$  sphere in  $\mathbb{R}^N$ . It is thus straightforward to extend the Theorem to matrices with

such rows as well. Furthermore, these projections are rotation invariant (often referred to as “universal” in CS systems), meaning that the theorem remains valid for sparse signals in any basis  $\Psi$ , i.e., for  $\mathbf{x}, \mathbf{s}$  belonging to  $\Sigma_{\Psi, K}^* := \{\mathbf{u} = \Psi\boldsymbol{\alpha} \in \mathbb{R}^N : \boldsymbol{\alpha} \in \Sigma_K^*\}$ . This is true since for any orthonormal basis  $\Psi \in \mathbb{R}^{N \times N}$ ,  $\Phi' = \Phi\Psi \sim \mathcal{N}^{M \times N}(0, 1)$  when  $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$ .

### C. Related Work

A similar result to Theorem 2 has been recently shown for sign measurements of non-sparse signals in the context of quantization using frame permutations [49]. Specifically, it has been shown that reconstruction from sign measurements of signals can be achieved (almost surely) with an error rate decay arbitrarily close to  $O(1/M)$ . Our main contribution here is demonstrating that this result is true uniformly for all  $K$ -sparse vectors in  $\mathbb{R}^N$ , given a *sparse* and *consistent* decoder. Our results, in addition to introducing the almost linear dependence on  $K$ , also show that proving this error bound uniformly for all  $K$ -sparse signals involves a logarithmic penalty in  $(MN)/K$ . This does not seem to be necessary from the lower bound in the previous section. We will see in Section V that for Gaussian matrices, the optimal error behavior is empirically exhibited on average. Finally, we note that for a constant  $\epsilon_o$ , the number of measurements required to guarantee (7) is  $M = O(K \log N)$ , nearly the same as order in conventional CS.

Furthermore, since the first appearance of our work, a bound on the achievable reconstruction error for compressible signals and for signals in arbitrary subsets of  $\mathbb{R}^N$  appeared in [31, 32]. Specifically for compressible signals, that works leads to error decay  $\epsilon = O((\frac{K}{M} \log \frac{N}{K})^{1/4})$ , which decreases more slowly (with  $K/M$ ) than both our bound and the one provided in [49]. However, the results in [31, 32] are for more general classes of signals. These works propose also convex optimization algorithms with provable reconstruction guarantees.

We can also view the binary measurements as a *hash* or a *sketch* of the signal. With this interpretation of the result we guarantee with high probability that no sparse vectors with Euclidean distance greater than  $\epsilon_o$  will “hash” to the same binary measurements. In fact, similar results play a key role in *locality sensitive hashing* (LSH), a technique that aims to efficiently perform approximate nearest neighbors searches from quantized projections [50–53]. Most LSH results examine the performance on point-clouds of a discrete number of signals instead of the infinite subspaces that we explore

in this paper. Furthermore, the primary goal of the LSH is to preserve the structure of the nearest neighbors with high probability. Instead, in this paper we are concerned with the ability to reconstruct the signal from the hash, as well as the robustness of this reconstruction to measurement noise and signal model mismatch. To enable these properties, we require a property of the mapping  $A$  that preserves the structure (geometry) of the entire signal set. Thus, in the next section we seek an embedding property of  $A$  that preserves geometry for the set of sparse signals and thus ensures robust reconstruction.

### III. ACQUISITION AND RECONSTRUCTION ROBUSTNESS

#### A. Binary $\epsilon$ -stable embeddings

In this section we establish an embedding property for the 1-bit CS mapping  $A$  that ensures that the sparse signal geometry is preserved in the measurements, analogous to the RIP for real-valued measurements. This robustness property enables us to upper bound the reconstruction performance even when some measurement signs have been changed due to noise. Conventional CS achieves robustness via the  $\delta$ -stable embeddings of sparse vectors (2) discussed in Section I. This embedding is a restricted *quasi-isometry* between the metric spaces  $(\mathbb{R}^N, d_X)$  and  $(\mathbb{R}^M, d_Y)$ , where the distance metrics  $d_X$  and  $d_Y$  are the  $\ell_2$ -norm in dimensions  $N$  and  $M$ , respectively, and the domain is restricted to sparse signals.<sup>4</sup> We seek a similar definition for our embedding; however, now the signals and measurements lie in the different spaces  $S^{N-1}$  and  $\mathcal{B}^M$ , respectively. Thus, we first consider appropriate distance metrics in these spaces.

The Hamming distance is the natural distance for counting the number of unequal bits between two measurement vectors. Specifically, for  $\bar{a}, \bar{b} \in \mathcal{B}^M$  we define the *normalized Hamming distance* as

$$d_H(\bar{a}, \bar{b}) = \frac{1}{M} \sum_{i=1}^M \bar{a}_i \oplus \bar{b}_i,$$

where  $\bar{a} \oplus \bar{b}$  is the XOR operation between  $\bar{a}, \bar{b} \in \mathcal{B}$  such that  $\bar{a} \oplus \bar{b}$  equals 0 if  $\bar{a} = \bar{b}$  and 1 otherwise. The distance is normalized such that  $d_H \in [0, 1]$ . In the signal

<sup>4</sup>A function  $A : X \rightarrow Y$  is called a *quasi-isometry* between metric spaces  $(X, d_X)$  and  $(Y, d_Y)$  if there exists  $C > 0$  and  $D \geq 0$  such that  $\frac{1}{C}d_X(\mathbf{x}, \mathbf{s}) - D \leq d_Y(A(\mathbf{x}), A(\mathbf{s})) \leq Cd_X(\mathbf{x}, \mathbf{s}) + D$  for  $\mathbf{x}, \mathbf{s} \in X$ , and  $E > 0$  such that  $d_Y(y, A(\mathbf{x})) < E$  for all  $y \in Y$  [54]. Since  $D = 0$  for  $\delta$ -stable embeddings, they are also called bi-Lipschitz mappings.

space we only consider unit-norm vectors, thus, a natural distance is the angle formed by any two of these vectors. Specifically, for  $\mathbf{x}, \mathbf{s} \in S^{N-1}$ , we consider

$$d_S(\mathbf{x}, \mathbf{s}) := \frac{1}{\pi} \arccos\langle \mathbf{x}, \mathbf{s} \rangle.$$

As with the Hamming distance, we normalize the true angle  $\arccos\langle \mathbf{x}, \mathbf{s} \rangle$  such that  $d_S \in [0, 1]$ . Note that since both vectors have the same norm, the inner product  $\langle \mathbf{x}, \mathbf{s} \rangle$  can easily be mapped to the  $\ell_2$ -distance using the polarization identity.

Using these distance metrics we define the binary stable embedding.

**Definition 1** (Binary  $\epsilon$ -Stable Embedding). *Let  $\epsilon \in (0, 1)$ . A mapping  $A : \mathbb{R}^N \rightarrow \mathcal{B}^M$  is a **binary  $\epsilon$ -stable embedding** (B $\epsilon$ SE) of order  $K$  for sparse vectors if*

$$d_S(\mathbf{x}, \mathbf{s}) - \epsilon \leq d_H(A(\mathbf{x}), A(\mathbf{s})) \leq d_S(\mathbf{x}, \mathbf{s}) + \epsilon$$

for all  $\mathbf{x}, \mathbf{s} \in S^{N-1}$  with  $|\text{supp}(\mathbf{x}) \cup \text{supp}(\mathbf{s})| \leq K$ .

Our definition describes a specific quasi-isometry between the two metric spaces  $(S^{N-1}, d_S)$  and  $(\mathcal{B}^M, d_H)$ , restricted to sparse vectors. While this mirrors the form of the  $\delta$ -stable embedding for sparse vectors, one important difference is that the sensitivity term  $\epsilon$  is additive, rather than multiplicative, and thus the B $\epsilon$ SE is not bi-Lipschitz. This is a necessary side-effect of the loss of information due to quantization.

Any B $\epsilon$ SE  $A(\cdot)$  of order  $2K$  enables robustness guarantees on any reconstruction algorithm extracting a unit sparse signal estimate  $\mathbf{x}^*$  of  $\mathbf{x} \in \Sigma_K^*$ . In this case, the angular error is immediately bounded by

$$d_S(\mathbf{x}, \mathbf{x}^*) \leq d_H(A(\mathbf{x}), A(\mathbf{x}^*)) + \epsilon.$$

Thus, if an algorithm returns a unit norm sparse solution with measurements that are not consistent (i.e.,  $d_H(A(\mathbf{x}), A(\mathbf{x}^*)) > 0$ ), as is the case with several algorithms [28–30], then the worst-case angular reconstruction error is close to Hamming distance between the estimate’s measurements’ signs and the original measurements’ signs. Section V verifies this behavior with simulation results. Furthermore, in Section III-C we use the B $\epsilon$ SE property to guarantee that if measurements are corrupted by noise or if signals are not exactly sparse, then the reconstruction error is bounded.

Note that, in the best case, for a B $\epsilon$ SE  $A(\cdot)$ , the angular error of any sparse and consistent  $\Delta^{\text{1bit}}(\bar{\mathbf{y}}, \Phi, K)$  decoder is bounded by  $\epsilon$  since then  $d_H(A(\mathbf{x}), A(\mathbf{x}^*)) = 0$ . As we have seen earlier this is to be expected because, unlike conventional noiseless CS, quantization fundamentally introduces uncertainty and exact recovery

cannot be guaranteed. This is an obvious consequence of the mapping of the infinite set  $\Sigma_K^*$  to a discrete set of quantized values.

We next identify a class of matrices  $\Phi$  for which  $A$  is a  $B\epsilon$ SE.

### B. Binary $\epsilon$ -stable embeddings via random projections

As is the case for conventional CS systems with RIP, designing a  $\Phi$  for 1-bit CS such that  $A$  has the  $B\epsilon$ SE property is possibly a computationally intractable task (and no such algorithm is yet known). Fortunately, an overwhelming number of “good” matrices do exist. Specifically we again focus our analysis on Gaussian matrices  $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$  as in as in Section II-B. As motivation that this choice of  $\Phi$  will indeed enable robustness, we begin with a classical concentration of measure result for binary measurements from a Gaussian matrix.

**Lemma 2.** *Let  $\mathbf{x}, \mathbf{s} \in S^{N-1}$  be a pair of arbitrary fixed vectors, draw  $\Phi$  according to  $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$ , and let the mapping  $A : \mathbb{R}^N \rightarrow \mathcal{B}^M$  be defined as in (3). Fix  $\epsilon > 0$ . Then we have*

$$\mathbb{P} \left( \left| d_H(A(\mathbf{x}), A(\mathbf{s})) - d_S(\mathbf{x}, \mathbf{s}) \right| \leq \epsilon \right) \geq 1 - 2e^{-2\epsilon^2 M}, \quad (8)$$

where the probability is with respect to the generation of  $\Phi$ .

*Proof:* This lemma is a simple consequence of Lemma 3.2 in [55] which shows that, for one measurement,  $\mathbb{P}[A_j(\mathbf{x}) \neq A_j(\mathbf{s})] = d_S(\mathbf{x}, \mathbf{s})$ . The result then follows by applying Hoeffding’s inequality to the binomial random variable  $Md_H(A(\mathbf{x}), A(\mathbf{s}))$  with  $M$  trials. ■

In words, Lemma 2 implies that the Hamming distance between two binary measurement vectors  $A(\mathbf{x}), A(\mathbf{s})$  tends to the angle between the signals  $\mathbf{x}$  and  $\mathbf{s}$  as the number of measurements  $M$  increases. In [55] this fact is used in the context of randomized rounding for max-cut problems; however, this property has also been used in similar contexts as ours with regards to preservation of inner products from binary measurements [56, 57].

The expression (8) indeed looks similar to the definition of the  $B\epsilon$ SE, however, it only holds for a fixed pair of arbitrary (not necessarily sparse) signals, chosen prior to drawing  $\Phi$ . Our goal is to extend (8) to cover the entire set of sparse signals. Indeed, concentration results similar to Lemma 2, although expressed in terms of norms, have been used to demonstrate the RIP [15].

These techniques usually demonstrate that the cardinality of the space of all sparse signals is sufficiently small, such that the concentration result can be applied to demonstrate that distances are preserved with relatively few measurements.

Unfortunately, due to the non-linearity of  $A$  we cannot immediately apply Lemma 2 using the same procedure as in [15]. To briefly summarize, [15] proceeds by covering the set of all  $K$ -sparse signals  $\Sigma_K$  with a finite set of points (with covering radius  $\delta > 0$ ). A concentration inequality is then applied to this set of points. Since any sparse signal lies in a  $\delta$ -neighborhood of at least one such point, the concentration property can be extended from the finite set to  $\Sigma_K$  by bounding the distance between the measurements of the points within the  $\delta$ -neighborhood. Such an approach cannot be used to extend (8) to  $\Sigma_K$ , because the severe discontinuity of our mapping does not permit us to characterize the measurements  $A(\mathbf{x} + \mathbf{s})$  using  $A(\mathbf{x})$  and  $A(\mathbf{s})$  and obtain a bound on the distance between measurements of signals in a  $\delta$ -neighborhood.

To resolve this issue, we extend Lemma 2 to include all points within Euclidean balls around the vectors  $\mathbf{x}$  and  $\mathbf{s}$  inside the (sub) sphere  $\Sigma^*(T) := \{\mathbf{u} \in S^{N-1} : \text{supp } \mathbf{u} \subset T\}$  for some fixed support set  $T \subset [N] := \{1, \dots, N\}$  of size  $|T| = D$ . Define the  $\delta$ -ball  $B_\delta(\mathbf{x}) := \{\mathbf{a} \in S^{N-1} : \|\mathbf{x} - \mathbf{a}\|_2 < \delta\}$  to be the ball of Euclidean distance  $\delta$  around  $\mathbf{x}$ , and let  $B_\delta^*(\mathbf{x}) := B_\delta(\mathbf{x}) \cap \Sigma^*(T)$ .

**Lemma 3.** *Given  $T \subset [N]$  of size  $|T| = D$ , let  $\Phi$  be a matrix generated as  $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$ , and let the mapping  $A : \mathbb{R}^N \rightarrow \mathcal{B}^M$  be defined as in (3). Fix  $\epsilon > 0$  and  $0 \leq \delta \leq 1$ . For any  $\mathbf{x}, \mathbf{s} \in \Sigma^*(T)$ , we have*

$$\mathbb{P} \left( \begin{aligned} &\forall \mathbf{u} \in B_\delta^*(\mathbf{x}), \forall \mathbf{v} \in B_\delta^*(\mathbf{s}), \\ &\left| d_H(A(\mathbf{u}), A(\mathbf{v})) - d_S(\mathbf{x}, \mathbf{s}) \right| \leq \epsilon + \sqrt{\frac{\pi}{2} D} \delta \\ &\geq 1 - 2e^{-2\epsilon^2 M}. \end{aligned} \right)$$

The proof of this result is given in Appendix D. It should be noted that the proof does not depend on the radial behavior of the Gaussian pdf in  $\mathbb{R}^N$ . In other words, this result is easily generalizable to matrices  $\Phi$  whose rows are independent random vectors drawn from an isotropic pdf.

In words, if the width  $\delta$  is sufficiently small, then the Hamming distance between the 1-bit measurements  $A(\mathbf{u}), A(\mathbf{v})$  of any points  $\mathbf{u}, \mathbf{v}$  within the balls  $B_\delta^*(\mathbf{x}), B_\delta^*(\mathbf{s})$ , respectively, will be close to the angle between the centers of the balls.

Lemma 3 is key for providing a similar argument to that in [15]. We now simply need to count the number



of pairs of  $K$ -sparse signals that are euclidean distance  $\delta$  apart. The Lemma can then be invoked to demonstrate that the angles between all of these pairs will be approximately preserved by our mapping.<sup>5</sup> Thus, with Lemma 3 under our belt, we demonstrate in Appendix E the following result.

**Theorem 3.** *Let  $\Phi$  be a matrix generated as  $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$  and let the mapping  $A : \mathbb{R}^N \rightarrow \mathcal{B}^M$  be defined as in (3). Fix  $0 \leq \eta \leq 1$  and  $\epsilon > 0$ . If the number of measurements is*

$$M \geq \frac{2}{\epsilon^2} (K \log(N) + 2K \log(\frac{35}{\epsilon}) + \log(\frac{2}{\eta})), \quad (9)$$

*then with probability exceeding  $1 - \eta$ , the mapping  $A$  is a BeSE of order  $K$  for sparse vectors.*

As with Lemma 3, the theorem extends easily to matrices  $\Phi$  with independent rows in  $\mathbb{R}^N$  drawn from an isotropic pdf in this space.

By choosing  $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$  with  $M = O(K \log N)$ , with high probability we ensure that the mapping  $A$  is a BeSE. Additionally, using (9) with a fixed  $\eta$  and the development in Appendix G, we find that the error decreases as

$$\epsilon = O\left(\sqrt{\frac{K}{M} \log \frac{MN}{K}}\right).$$

Unfortunately, this decay rate is slower, roughly by a factor of  $\sqrt{K/M}$ , than the lower bound in Section II-A. This error rate results from an application of the Chernoff-Hoeffding inequality in the proof of Theorem 3. An open question is whether it is possible to obtain a tighter bound (with optimal error rate) for this robustness property.

As with Theorem 2, Gaussian matrices provide a universal mapping, i.e., the result remains valid for sparse signals in a basis  $\Psi \in \mathbb{R}^{N \times N}$ . Moreover, Theorem 3 can also be extended to rows of  $\Phi$  that are drawn uniformly on the sphere, since the rows of  $\Phi$  in Theorem 3 can be normalized without affecting the outcome of the proof. Note that normalizing the Gaussian rows of  $\Phi$  is as if they had been drawn from a uniform distribution of unit-norm signals.

We have now established a random construction providing robust BeSEs with high probability: 1-bit quantized Gaussian projections. We now make use of this robustness by considering an example where the measurements are corrupted by Gaussian noise.

<sup>5</sup> We note that the covering argument in the proof of Theorem 2 also employs  $\delta$ -balls in similar fashion but only considers the probability that  $d_H = 0$ , rather than the concentration inequality.

### C. Noisy measurements and compressible signals

In practice, hardware systems may be inaccurate when taking measurements; this is often modeled by additive noise. The mapping  $A$  is robust to noise in an unusual way. After quantization, the measurements can only take the values  $-1$  or  $1$ . Thus, we can analyze the reconstruction performance from corrupted measurements by considering how many measurements flip their signs. For example, we analyze the specific case of Gaussian noise on the measurements prior to quantization, i.e.,

$$A_n(\mathbf{x}) := \text{sign}(\Phi \mathbf{x} + \mathbf{n}), \quad (10)$$

where  $\mathbf{n} \in \mathbb{R}^M$  has i.i.d. elements  $n_i \sim \mathcal{N}(0, \sigma^2)$ . In this case, we demonstrate, via the following lemma, a bound on the Hamming distance between the corrupted and ideal measurements with the BeSE from Theorem 3 (see Appendix F).

**Lemma 4.** *Let  $\Phi$  be a matrix generated as  $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$ , let the mapping  $A : \mathbb{R}^N \rightarrow \mathcal{B}^M$  be defined as in (3), and let  $A_n : \mathbb{R}^N \rightarrow \mathcal{B}^M$  be defined as in (10). Let  $\mathbf{n} \in \mathbb{R}^M$  be a Gaussian random vector with i.i.d. components  $n_i \sim \mathcal{N}(0, \sigma^2)$ . Fix  $\gamma > 0$ . Then, given  $\mathbf{x} \in \mathbb{R}^N$ , we have*

$$\begin{aligned} \mathbb{E}(d_H(A_n(\mathbf{x}), A(\mathbf{x}))) &\leq e(\sigma, \|\mathbf{x}\|_2), \\ \mathbb{P}(d_H(A_n(\mathbf{x}), A(\mathbf{x})) > e(\sigma, \|\mathbf{x}\|_2) + \gamma) &\leq e^{-2M\gamma^2}, \end{aligned}$$

where  $e(\sigma, \|\mathbf{x}\|_2) := \frac{1}{2} \frac{\sigma}{\sqrt{\|\mathbf{x}\|_2^2 + \sigma^2}} \leq \frac{1}{2} \frac{\sigma}{\|\mathbf{x}\|_2}$ .

If  $\mathbf{x}_n^*$  is the estimate from a sparse consistent reconstruction decoder  $\Delta^{\text{1bit}}(A_n(\mathbf{x}), \Phi, K)$  from the measurements  $A_n(\mathbf{x})$  with  $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$  and if  $M$  satisfies (9), then it immediately follows from Lemma 4 and Theorem 3 that

$$\begin{aligned} d_S(\mathbf{x}_n^*, \mathbf{x}) &\leq d_H(A_n(\mathbf{x}), A(\mathbf{x})) + \epsilon \\ &\leq \frac{1}{2} \frac{\sigma}{\|\mathbf{x}\|_2} + \gamma + \epsilon, \end{aligned} \quad (11)$$

with a probability higher than  $1 - e^{-2M\gamma^2} - \eta$ . Given alternative noise distributions, e.g., Poisson noise, a similar analysis can be carried out to determine the likely number of sign flips and thus provide a bound on the error due to noise.

Another practical consideration is that real signals are not always strictly  $K$ -sparse. Indeed, it may be the case that signals are *compressible*; i.e., they can be closely approximated by a  $K$ -sparse signal. In this case, we can reuse the non-uniform result of Lemma 2 to see that, given  $\mathbf{x} \in \mathbb{R}^N$  and for  $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$ ,

$$\mathbb{P}(d_H(A(\mathbf{x}), A(\mathbf{x}_K)) > d_S(\mathbf{x}, \mathbf{x}_K) + \gamma) \leq e^{-2M\gamma^2}.$$

In similar fashion to (11), if  $M$  satisfies (9), this result and Theorem 3 imply that, given  $\mathbf{x} \in S^{N-1}$  (not necessarily sparse) and for  $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$  the angular reconstruction error of  $\mathbf{x}^* = \Delta^{\text{1bit}}(A(\mathbf{x}), \Phi, K)$  is such that  $d_S(\mathbf{x}^*, \mathbf{x}_K) \leq d_H(A(\mathbf{x}^*), A(\mathbf{x}_K)) + \epsilon = d_H(A(\mathbf{x}), A(\mathbf{x}_K)) + \epsilon \leq d_S(\mathbf{x}, \mathbf{x}_K) + \gamma + \epsilon$ , with probability higher than  $1 - e^{-2M\gamma^2} - \eta$ . Therefore, from the triangular inequality on  $d_S$ , this provides the bound

$$d_S(\mathbf{x}^*, \mathbf{x}) \leq 2d_S(\mathbf{x}, \mathbf{x}_K) + \gamma + \epsilon,$$

with the same probability. Much like conventional CS results, the reconstruction error depends on the magnitude of the best  $K$ -term approximation error of the signal, here expressed angularly by  $d_S(\mathbf{x}, \mathbf{x}_K)$ .

This reconstruction error bound is non-uniform with respect to the selection of  $\mathbf{x} \in \mathbb{R}^N$ . A uniform bound on the BeSE for more general classes of signals is developed in [31, 32], albeit with a worse error decay— $\epsilon = O((\frac{K}{M} \log \frac{N}{K})^{1/4})$  for compressible signals.

Thus far we have demonstrated a lower bound on the reconstruction error from 1-bit measurements (Theorem 2) and introduced a condition on the mapping  $A$  that enables stable reconstruction in noiseless, noisy, and compressible settings (Definition 1). We have furthermore demonstrated that a large class of random matrices—specifically matrices with coefficients drawn from a Gaussian distribution and matrices with rows drawn uniformly from the unit sphere—provide good mappings (Theorem 3).

Using these results we can characterize the error performance of any algorithm that reconstructs a  $K$ -sparse signal. If the reconstructed signal quantizes to the same quantization point as the original data, then the error is characterized by Theorem 2. If the algorithm terminates unable to reconstruct a signal consistent with the quantized data, then Theorem 3 describes how far the solution is from the original signal. Since (R<sub>1BCS</sub>) is a combinatorially complex problem, in the next section we describe a new greedy reconstruction algorithm that attempts to find a solution as consistent with the measurements as possible, while guaranteeing this solution is  $K$ -sparse.

#### IV. BIHT: A SIMPLE FIRST-ORDER RECONSTRUCTION ALGORITHM

##### A. Problem formulation and algorithm definition

We now introduce a simple algorithm for the reconstruction of sparse signals from 1-bit compressive measurements. Our algorithm, *Binary Iterative Hard Thresh-*

*olding* (BIHT), is a simple modification of IHT, the real-valued algorithm from which it takes its name [14]. Demonstrating theoretical convergence guarantees for BIHT is a subject of future work (and thus not shown in this paper), however the algorithm is of significant value since it *i)* has a simple and intuitive formulation and *ii)* outperforms previous algorithms empirically, demonstrated in Section V. We further note that the IHT algorithm has recently been extended to handle measurement non-linearities [58]; however, these results do not apply to quantized measurements since quantization does not satisfy the requirements in [58].

We briefly recall that the IHT algorithm consists of two steps that can be interpreted as follows. The first step can be thought of as a gradient descent to reduce the least squares objective  $\|\mathbf{y} - \Phi\mathbf{x}\|_2^2/2$ . Thus, at iteration  $l$ , IHT proceeds by setting  $\mathbf{a}^{l+1} = \mathbf{x}^l + \Phi^T(\mathbf{y} - \Phi\mathbf{x}^l)$ . The second step imposes a sparse signal model by projecting  $\mathbf{a}^{l+1}$  onto the “ $\ell_0$  ball”, i.e., selecting the  $K$  largest in magnitude elements. Thus, IHT for CS can be thought of as trying to solve the problem

$$\underset{\mathbf{u}}{\operatorname{argmin}} \frac{1}{2} \|\mathbf{y} - \Phi\mathbf{u}\|_2^2 \quad \text{s.t.} \quad \|\mathbf{u}\|_0 = K. \quad (12)$$

The BIHT algorithm simply modifies the first step of IHT to instead minimize a consistency-enforcing objective. Specifically, given an initial estimate  $\mathbf{x}^0 = \mathbf{0}$  and 1-bit measurements  $\bar{\mathbf{y}}$ , at iteration  $l$  BIHT computes

$$\mathbf{a}^{l+1} = \mathbf{x}^l + \frac{\tau}{2} \Phi^T(\bar{\mathbf{y}} - A(\mathbf{x}^l)), \quad (13)$$

$$\mathbf{x}^{l+1} = \eta_K(\mathbf{a}^{l+1}), \quad (14)$$

where  $A$  is defined as in (3),  $\tau$  is a scalar that controls gradient descent step-size, and the function  $\eta_K(\mathbf{v})$  computes the best  $K$ -term approximation of  $\mathbf{v}$  by thresholding. Once the algorithm has terminated (either consistency is achieved or a maximum number of iterations have been reached), we then normalize the final estimate to project it onto the unit sphere. Section IV-B discusses several variations of this algorithm, each with different properties.

The key to understanding BIHT lies in the formulation of the objective. The following Lemma shows that the term  $\Phi^T(\bar{\mathbf{y}} - A(\mathbf{x}^l))$  in (13) is in fact the negative subgradient of a convex objective  $\mathcal{J}$ . Let  $[\cdot]_-$  denote the negative function, i.e.,  $([\mathbf{u}]_-)_i = [u_i]_-$  with  $[u_i]_- = u_i$  if  $u_i < 0$  and 0 else, and  $\mathbf{u} \odot \mathbf{v}$  denote the Hadamard product, i.e.,  $(\mathbf{u} \odot \mathbf{v})_i = u_i v_i$  for two vectors  $\mathbf{u}$  and  $\mathbf{v}$ .

**Lemma 5.** *The quantity  $\frac{1}{2} \Phi^T(A(\mathbf{x}) - \bar{\mathbf{y}})$  in (13) is a subgradient of the convex one-sided  $\ell_1$ -norm*

$$\mathcal{J}(\mathbf{x}) = \|[\bar{\mathbf{y}} \odot (\Phi\mathbf{x})]_-\|_1,$$

Thus, BIHT aims to decrease  $\mathcal{J}$  at each step (13).

*Proof:* We first note that  $\mathcal{J}$  is convex. We can write  $\mathcal{J}(\mathbf{x}) = \sum_i \mathcal{J}_i(\mathbf{x})$  with each convex function  $\mathcal{J}_i$  given by

$$\mathcal{J}_i(\mathbf{x}; \bar{\mathbf{y}}, \Phi) = \begin{cases} |\langle \boldsymbol{\varphi}_i, \mathbf{x} \rangle|, & \text{if } A_i(\mathbf{x}) \bar{y}_i < 0, \\ 0, & \text{else,} \end{cases}$$

where  $\boldsymbol{\varphi}_i$  denotes a row of  $\Phi$  and  $A_i(\mathbf{x}) = \text{sign} \langle \boldsymbol{\varphi}_i, \mathbf{x} \rangle$ . Moreover, if  $\langle \boldsymbol{\varphi}_i, \mathbf{x} \rangle \neq 0$ , then the gradient of  $\mathcal{J}_i$  is

$$\begin{aligned} \nabla \mathcal{J}_i(\mathbf{x}; \bar{\mathbf{y}}, \Phi) &= \frac{1}{2} (A_i(\mathbf{x}) - \bar{y}_i) \boldsymbol{\varphi}_i \\ &= \begin{cases} A_i(\mathbf{x}) \boldsymbol{\varphi}_i & \text{if } \bar{y}_i A_i(\mathbf{x}) < 0, \\ 0, & \text{else,} \end{cases} \end{aligned}$$

while if  $\langle \boldsymbol{\varphi}_i, \mathbf{x} \rangle = 0$ , then the gradient is replaced by the subdifferential set

$$\begin{aligned} \nabla \mathcal{J}_i(\mathbf{x}; \bar{\mathbf{y}}, \Phi) &= \left\{ \frac{\xi}{2} (A_i(\mathbf{x}) - \bar{y}_i) \boldsymbol{\varphi}_i : \xi \in [0, 1] \right\} \\ &\ni \frac{1}{2} (A_i(\mathbf{x}) - \bar{y}_i) \boldsymbol{\varphi}_i. \end{aligned}$$

Thus, by summing over  $i$  we conclude that  $\frac{1}{2} \Phi^T (A(\mathbf{x}) - \bar{\mathbf{y}}) \in \nabla \mathcal{J}(\mathbf{x}; \bar{\mathbf{y}}, \Phi)$ . ■

Consequently, the BIHT algorithm can be thought of as trying to solve the problem:

$$\mathbf{x}^* = \underset{\mathbf{u}}{\text{argmin}} \|\bar{\mathbf{y}} \odot (\Phi \mathbf{u})\|_1 \text{ s.t. } \|\mathbf{u}\|_0 = K, \|\mathbf{u}\|_2 = 1.$$

Observe that since  $\bar{\mathbf{y}} \odot (\Phi \mathbf{x})$  simply scales the elements of  $\Phi \mathbf{x}$  by the signs  $\bar{\mathbf{y}}$ , minimizing the one-sided  $\ell_1$  objective enforces a positivity requirement,

$$\bar{\mathbf{y}} \odot (\Phi \mathbf{x}) \geq 0, \quad (15)$$

that, when satisfied, implies consistency.

Previously proposed 1-bit CS algorithms have used a one-sided  $\ell_2$ -norm to impose consistency [27–30]. Specifically, they have applied a constraint or objective that takes the form  $\|\bar{\mathbf{y}} \odot (\Phi \mathbf{x})\|_2^2 / 2$ . Both the one-sided  $\ell_1$  and  $\ell_2$  functions imply a consistent solution when they evaluate to zero, and thus, both approaches are capable of enforcing consistency. However, the choice of the  $\ell_1$  vs.  $\ell_2$  penalty term makes a significant difference in performance depending on the noise conditions. We explore this difference in the experiments in Section V.

## B. BIHT shifts

Several modifications can be made to the BIHT algorithm that may improve certain performance aspects, such as consistency, reconstruction error, or convergence speed. While a comprehensive comparison is beyond the scope of this paper, we believe that such variations

exhibit interesting and useful properties that should be mentioned.

**Projection onto sphere at each iteration.** We can enforce that every intermediate solution have unit  $\ell_2$  norm. To do this, we modify the ‘‘impose signal model’’ step (14) by normalizing after choosing the best  $K$ -term approximation, i.e., we replace the update of  $\mathbf{x}^{l+1}$  in (14) by  $\mathbf{x}^{l+1} = U(\eta_K(\mathbf{a}^{l+1}))$ , where  $U(\mathbf{v}) = \mathbf{v} / \|\mathbf{v}\|_2$ . While this step is found in previous algorithms such as [28–30], empirical observations suggest that it not required for BIHT to converge to an appropriate solution.

If we choose to impose the projection,  $\Phi$  must be appropriately normalized or, equivalently, the step size of the gradient descent must be carefully chosen. Otherwise, the algorithm will not converge. Empirically, we have found that for a Gaussian matrix, an appropriate scaling is  $1/(\sqrt{M} \|\Phi\|_2)$ , where the  $1/\|\Phi\|_2$  controls the amplification of the estimate from  $\Phi^T$  in the gradient descent step (13) and the  $1/\sqrt{M}$  ensures that  $\|\bar{\mathbf{y}} - A(\mathbf{x}^l)\|_2 \leq 2$ . Similar gradient step scaling requirements have been imposed in the conventional IHT algorithm and other sparse recovery algorithms as well (e.g., [9]).

**Minimizing hinge loss.** The one-sided  $\ell_1$ -norm is related to the *hinge-loss* function in the machine learning literature, which is known for its robustness to outliers [59]. Binary classification algorithms seek to enforce the same consistency function as in (15) by minimizing a function  $\sum_i [\kappa - \bar{y}_i (\Phi \mathbf{x})_i]_+ = \|\kappa \mathbf{1} - \bar{\mathbf{y}} \odot (\Phi \mathbf{x})\|_1$ , where  $[\cdot]_+$  sets negative elements to zero. When  $\kappa > 0$ , the objective is both convex and has a non-trivial solution. Further connections and interpretations are discussed in Section V. Thus, rather than minimizing the one-sided  $\ell_1$  norm, we can instead minimize the hinge-loss. The hinge-loss can be interpreted as ensuring that the minimum value that an unquantized measurement  $(\Phi \mathbf{x})_i$  can take is bounded away from zero, i.e.,  $|(\Phi \mathbf{x})_i| \geq \kappa$ . This requirement is similar to the sphere constraint in that it avoids a trivial solution; however, will perform differently than the sphere constraint. In this case, in the gradient descent step (13), we instead compute

$$\mathbf{a}^{l+1} = \mathbf{x}^l - \tau \Theta^T (\text{sign}(\Theta \mathbf{x}^l - \kappa) - 1) / 2$$

where  $\Theta = (\bar{\mathbf{y}} \odot \Phi)$  scales the rows of  $\Phi$  by the signs of  $\bar{\mathbf{y}}$ . Again, the step size must be chosen appropriately, this time as  $C_\kappa / \|\Phi\|_2$ , where  $C_\kappa$  is a parameter that depends on  $\kappa$ .

**General one-sided objectives.** In general, any function  $\mathcal{R}(\mathbf{x}) = \sum \mathcal{R}_i(x_i)$ , where  $\mathcal{R}_i$  is continuous and has a negative gradient for  $x_i \leq 0$  and is 0 for  $x_i > 0$ , can be used to enforce consistency. To employ such functions,

we simply compute the gradient of  $\mathcal{R}$  and apply it in (13). As an example, the previously mentioned one-sided  $\ell_2$ -norm has been used to enforce consistency in several algorithms. We can use it in BIHT by computing

$$\mathbf{a}^l = \mathbf{x}^l + \tau \Phi^T [\bar{\mathbf{y}} \odot \Phi \mathbf{x}^l]_+$$

in (13). We compare and contrast the behavior of the one-sided  $\ell_1$  and  $\ell_2$  norms in Section V.

## V. EXPERIMENTS

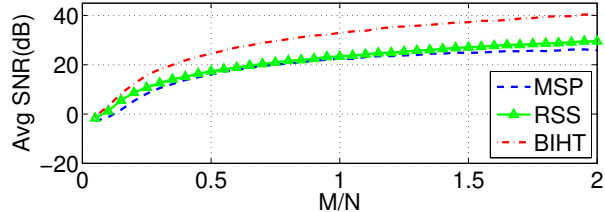
In this section we explore the performance of the BIHT algorithm and compare it to the performance of previous 1-bit CS algorithms. To make the comparison as straightforward as possible, we reproduced the experiments of [30] with the BIHT algorithm.

The experimental setup is as follows. For each data point, we draw a length- $N$ ,  $K$ -sparse signal with the non-zero entries drawn uniformly at random on the unit sphere, and we draw a new  $M \times N$  matrix  $\Phi$  with each entry  $\phi_{ij} \sim \mathcal{N}(0, 1)$ . We then compute the binary measurements  $\bar{\mathbf{y}}$  according to (3). Reconstruction of  $\mathbf{x}^*$  is performed from  $\bar{\mathbf{y}}$  with three algorithms: *matching sign pursuit* (MSP) [29], *restricted-step shrinkage* (RSS) [30], and BIHT (this paper); the algorithms will be depicted by dashed, dotted, and triangle lines, respectively. Each reconstruction in this setup is repeated for 1000 trials and with a fixed  $N = 1000$  and  $K = 10$  unless otherwise noted. Furthermore, we perform the trials for  $M/N$  within the range  $[0, 2]$ . Note that when  $M/N > 1$ , we are acquiring more measurements than the ambient dimension of the signal. While the  $M/N > 1$  regime is not interesting in conventional CS, it may be very practical in 1-bit systems that can acquire sign measurements at extremely high, super-Nyquist rates.

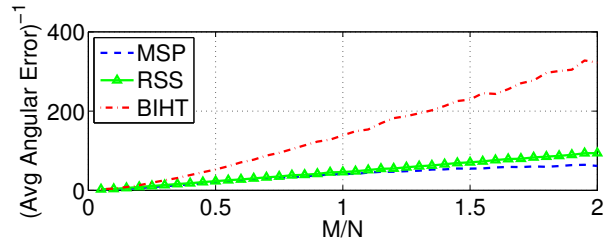
**Average error.** We begin by measuring the average reconstruction angular error  $\epsilon_{\text{sim}} := d_S(\mathbf{x}, \mathbf{x}^*)$  over the 1000 trials. Figure 2 displays the results of this experiment in two different ways: (i) the signal-to-noise ratio (SNR)<sup>6</sup> in Figure 2(a), to demonstrate that the performance of these techniques is practical (since the angular error is unintuitive to most observers), and (ii) the inverse of the angular error, i.e.,  $\epsilon_{\text{sim}}^{-1}$  in Figure 2(b), to compare with the performance predicted by Theorem 2.

We begin by comparing the performance of the algorithms. While we can observe that the angular error of each algorithm follows the same trend, BIHT

<sup>6</sup>In this paper we define the reconstruction SNR in decibels as  $\text{SNR}(\mathbf{x}) := 10 \log_{10}(\|\mathbf{x}\|_2^2 / \|\mathbf{x} - \mathbf{x}^*\|_2^2)$ . Note that this metric uses the standard euclidean error and not angular error.



(a)



(b)

Fig. 2: Average reconstruction angular error  $\epsilon_{\text{sim}}$  vs.  $M/N$ , plotted two ways. (a) SNR in decibels, and (b) Inverse angular error  $\epsilon_{\text{sim}}^{-1}$ . The plot demonstrates that BIHT yields a considerable improvement in reconstruction error, achieving an SNR as high as 40dB when  $M/N = 2$ . Furthermore, we see that the error behaves according  $\epsilon_{\text{sim}}^{-1} = O(M)$ , implying that on average we achieve the optimal performance rate given in Theorem 1.

obtains smaller error (or higher SNR) than the others, significantly so when  $M/N$  is greater than 0.35. The discrepancy in performance could be due to difference in the algorithms themselves, or perhaps, differences in their formulations for enforcing consistency. This is explored later in this section.

We now consider the actual performance trend. We see from Figure 2(b) that, above  $M/N = 0.35$  each line appears fairly linear, albeit with a different slope, implying that with all other variables fixed,  $\epsilon_{\text{sim}} = O(1/M)$ . This is on the order of the optimal performance as given by the bound given in Theorem 1 and predicted by Theorem 2 for Gaussian matrices.

**Consistency.** We also expose the relationship between the Hamming distance  $d_H(A(\mathbf{x}), A(\mathbf{x}^*))$  between the measurements of the true and reconstructed signal and the angular error of the true and reconstructed signal. Figure 3 depicts the Hamming distance vs. angular error for three different values of  $M/N$ . The particularly striking result is that BIHT returns significantly more consistent reconstructions than the two other algorithms<sup>7</sup>. This is clear from the fact that most of the red (plus) points lie on the y-axis while the majority of blue (dot)

<sup>7</sup>We observed this effect for ratio  $M/N$  as small as  $M/N = 0.1$  (not represented here).

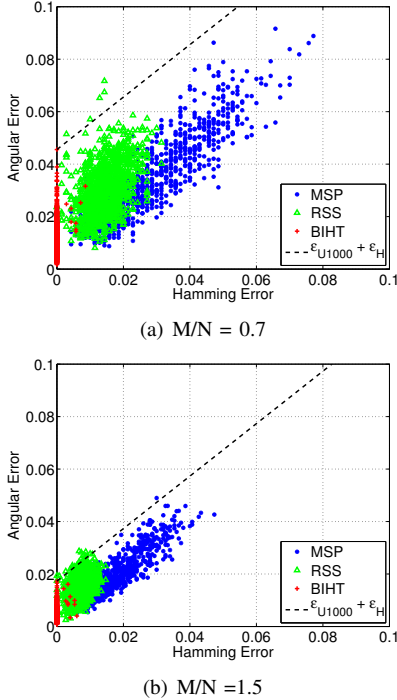


Fig. 3: Reconstruction angular error  $\epsilon_{\text{sim}}$  vs. measurement Hamming error  $\epsilon_H$ . BIHT returns a consistent solution in most trials. For sufficiently large  $M/N$  regimes, we see a linear relationship  $\epsilon_{\text{sim}} \approx C + \epsilon_H$  between the average angular error  $\epsilon_{\text{sim}}$  and the hamming error  $\epsilon_H$  where  $C$  is constant (see (a) and (b)). The B $\epsilon$ SE formulation in Definition 1 predicts that the angular error is bounded by the hamming error  $\epsilon_H$  in addition to an offset  $c$ . The dashed line  $\epsilon_{U1000} + \epsilon_H$  denotes the empirical upper bound for 1000 trials.

or green (triangle) points do not. We find that, even in significantly “under-sampled” regimes like  $M/N = 0.1$ , where the B $\epsilon$ SE is unlikely to hold, BIHT is likely to return a consistent solution (albeit with high variance of angular errors). We also find that in “over-sampled” regimes such as  $M/N = 1.7$ , the range of angular errors on the y-axis is small. Indeed, the range of angular errors shrinks as  $M/N$  increases, implying an imperical tightening of the B $\epsilon$ SE upper and lower bounds.

We can infer an interesting performance trend from Figures 3(a) and (b), where the B $\epsilon$ SE property may hold. Since the RSS and MSP algorithms often do not return a consistent solution, we can visualize the relationship between angular error and hamming error. Specifically, on average the angular reconstruction error is a linear function of hamming error,  $\epsilon_H = d_H(A(\mathbf{x}), A(\mathbf{x}^*))$ , as similarly expressed by the reconstruction error bound provided by B $\epsilon$ SE. Furthermore, if we let  $\epsilon_{1000}$  be the largest angular error (with consistent measurements) over 1000 trials, then we can suggest an empirical upper

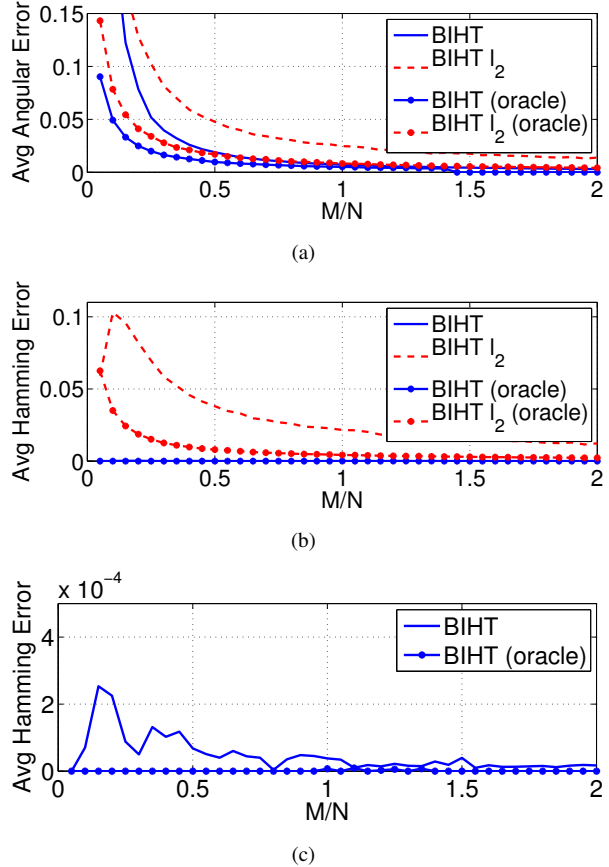


Fig. 4: Enforcing consistency: One-sided  $\ell_1$  vs. one-sided  $\ell_2$  BIHT. When BIHT attempts to minimize a one-sided  $\ell_2$  instead of a one-sided  $\ell_1$  objective, the performance significantly decreases. We find this to be the case even when an oracle provides the true signal support a priori. Note: (c) is simply a zoomed version (b).

bound for BIHT of  $\epsilon_{1000} + \epsilon_H$ . This upper bound is denoted by the dashed line in Figures 3(a) and (b).

**One-sided  $\ell_1$  vs. one-sided  $\ell_2$  objectives.** As demonstrated in Figures 2 and 3, the BIHT algorithm achieves significantly improved performance over MSP and RSS in both angular error and Hamming error (consistency). A significant difference between these algorithms and BIHT is that MSP and RSS seek to impose consistency via a one-sided  $\ell_2$ -norm, as described in Section IV-B. Minimizing either the one-sided  $\ell_1$  or one-sided  $\ell_2$  objectives will enforce consistency on the measurements of the solution; however, the behavior of these two terms appears to be significantly different, according to the previously discussed experiments.

To test the hypothesis that this term is the key differentiator between the algorithms, we implemented BIHT- $\ell_2$ , a one-sided  $\ell_2$  variation of the BIHT algo-

rithm that enabled a fair comparison of the one-sided objectives (see Section IV-B for details). We compared both the angular error and Hamming error performance of BIHT and BIHT- $\ell_2$ . Furthermore, we implemented *oracle assisted* variations of these algorithms where the true support of the signal is given a priori, i.e.,  $\eta_K$  in (14) is replaced by an operator that always selects the true support, and thus the algorithm only needs to estimate the correct coefficient values. The oracle assisted case can be thought of as a “best performance” bound for these algorithms. Using these algorithms, we perform the same experiment detailed at the beginning of the section.

The results are depicted in Figure 4. The angular error behavior of BIHT- $\ell_2$  is very similar to that of MSP and RSS and underperforms when compared to BIHT. We see the same situation with regards to Hamming error: BIHT finds consistent solutions for the majority of trials, but BIHT- $\ell_2$  does not. Thus, the results of this simulation suggest that the one-sided term plays a significant role in the quality of the solution obtained.

One way to explain the performance discrepancy between the two objectives comes from observing the connection between our reconstruction problem and binary classification. As explained previously, in the classification context, the one-sided  $\ell_1$  objective is similar to the hinge-loss, and furthermore, the one-sided  $\ell_2$  objective is similar to the so-called *square-loss*. Previous results in machine learning have shown that for typical convex loss functions, the minimizer of the hinge loss has the tightest bound between expected risk and the Bayes optimal solution [60] and good error rates, especially when considering robustness to outliers [60, 61]. Thus, the hinge loss is often considered superior to the square loss for binary classification.<sup>8</sup> One might suspect that since the one-sided  $\ell_1$ -objective is very similar to the hinge loss, it too should outperform other objectives in our context. Understanding why in our context, the geometry of the  $\ell_1$  and  $\ell_2$  objectives results in different performance is an interesting open problem.

We probed the one-sided  $\ell_1/\ell_2$  objectives further by testing the two versions of BIHT on noisy measurements. We flipped a number of measurement signs at random in each trial. For this experiment,  $N = M = 1000$  and  $K = 10$  are fixed, and we performed 100 trials. We varied the number of sign flips between 0% and 5% of the measurements. The results of the experiment are depicted in Figure 5. We see that for both the angular error

<sup>8</sup>Additional “well-behaved” loss functions (e.g., the Huber-ized hinge loss) have been proposed [62] and a host of classification algorithms related to this problem exist [61–65], both of which may prove useful in the 1-bit CS framework in the future.

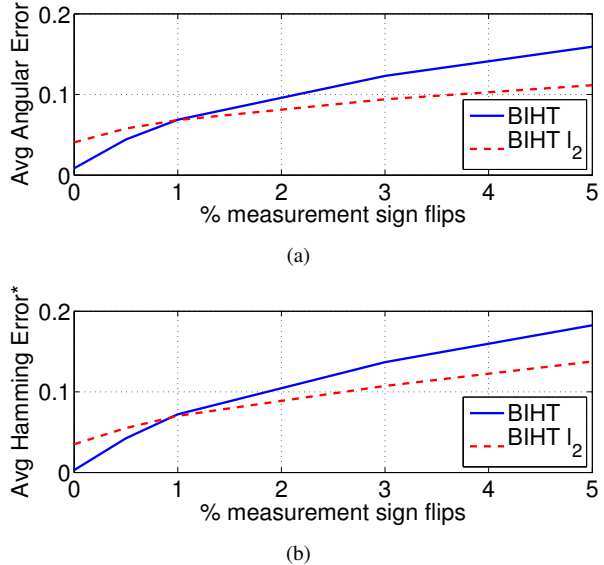


Fig. 5: Enforcing consistency with noise: One-sided  $\ell_1$  vs. one-sided  $\ell_2$  BIHT. When BIHT attempts to minimize a one-sided  $\ell_2$  instead of the one-sided  $\ell_1$  objective, the algorithm is more robust to flips of measurement signs. \*Note that the Hamming error in (b) is measured with regards to the *noisy* measurements, e.g., a Hamming error of zero means that we reconstructed the signs of the noisy measurements exactly.

in Figure 5(a) and Hamming error in Figure 5(b), that the one-sided  $\ell_1$  objective performs better when there are only a few errors and the one-sided  $\ell_2$  objective performs better when there are significantly more errors. This is expected since the  $\ell_1$  objective promotes sparse errors. This experiment implies that BIHT- $\ell_2$  (and the other one-sided  $\ell_2$ -based algorithms) may be more useful when the measurements contain significant noise that might cause a large number of sign flips, such as Gaussian noise.

**Performance with a fixed bit-budget.** In some applications we are interested in reducing the total number of bits acquired due to storage or communication costs. Thus, given a fixed total number of bits, an interesting question is how well 1-bit CS performs in comparison to conventional CS quantization schemes and algorithms. For the sake of brevity, we give a simple comparison here between the 1-bit techniques and uniform quantization with *Basis Pursuit DeNoising* (BPDN) [8] reconstruction. While BPDN is not the optimal reconstruction technique for quantized measurements, it (and its variants such as the LASSO [62]) is considered a benchmark technique for reconstruction from measurements with noise and furthermore, is widely used in practice.

The experiment proceeds as follows. Given a total number of bits and a (uniform) quantization bit-depth

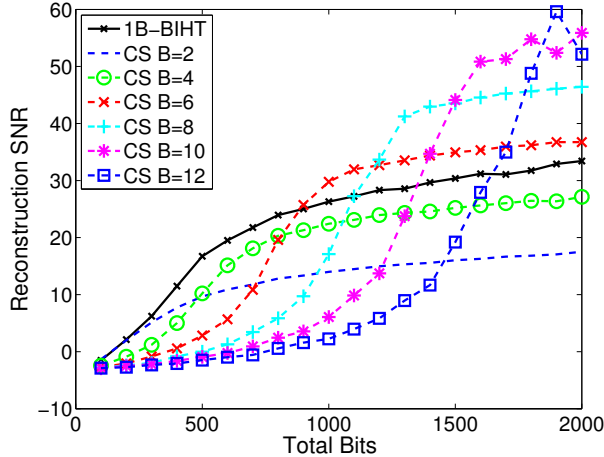


Fig. 6: Comparison of BIHT to conventional CS multibit uniform scalar quantization (multibit reconstructions performed using BPDN [8]). BIHT is competitive with standard CS working with multibit measurements when the total number of bits is severely constrained. In particular, the BIHT algorithm performs strictly better than CS with 4 bits per measurement.

$B$  (i.e., number of bits per measurement), we choose the number of measurements as  $M = \text{total bits}/B$ ,  $N = 2000$ , and the sparsity  $K = 20$ . The remainder of the experiment proceeds as described earlier (in terms of drawing matrices and signals). For bit depth greater than 1, we reconstruct using BPDN with an optimal choice of noise parameter and we scale the quantizer to such that signal can take full advantage of its dynamic range.

The results of this experiment are depicted in Figure 6. We see a common trend in each line: lackluster performance until “sufficient” measurements are acquired, then a slow but steady increase in performance as additional measurement are added, until a performance plateau is reached. Thus, since lower bit-depth implies that a larger number of measurements will be used, 1-bit CS reaches the performance plateau earlier than in the multi-bit case (indeed, the transition point is achieved at a higher number of total bits as the bit-depth is increased). This enables significantly improved performance when the rate is severely constrained and higher bit-rates per measurements would significantly reduce the number of available measurements. For higher bit-rates, as expected from the analysis in [35], using fewer measurements with refined quantization achieves better performance.

It is also important to note that, regardless of trend, the BIHT algorithm performs strictly better than BPDN with 4 bits per measurement and uniform quantization for the parameters tested here. This gain is consistent with similar gains observed in [28, 29]. A more thorough

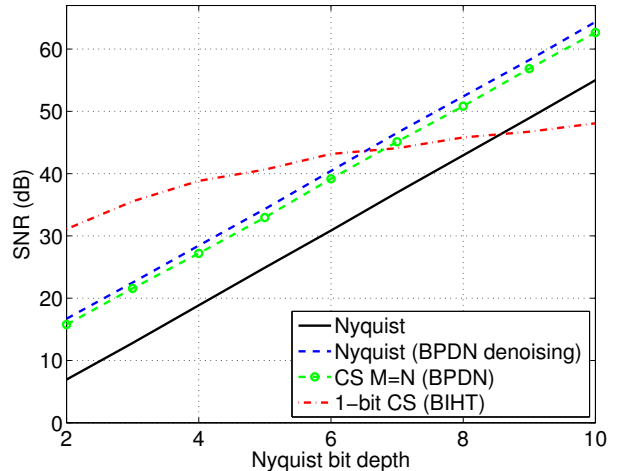


Fig. 7: Comparison of uniformly quantized Nyquist-rate samples with linear reconstruction (solid) and BPDN denoising (dashed), CS with  $M = N$  and BPDN reconstruction (dash-circle), and 1-bit quantized CS measurements with BIHT reconstruction (dash-dotted). Nyquist samples were quantized with bit-depth  $\beta \in [2, 10]$  and 1-bit CS used  $M = \beta N$  measurements; the same number of bits is used in each reconstruction. The Nyquist-rate lines have the classical 6.02dB/bit-depth slope, as expected. For a fixed number of bits, 1-bit CS does *not* follow this slope and outperforms conventional quantization when  $\beta < 6$ .

comparison of additional CS quantization techniques with 1-bit CS is a subject for future study.

**Comparison to quantized Nyquist samples.** In our final experiment, we compare the performance of the 1-bit CS technique to the performance of a conventional uniform quantizer applied to uniform Nyquist-rate samples. Specifically, in each trial we draw a new Nyquist-sampled signal in the same way as in our previous experiments and with fixed  $N = 2000$  and  $K = 20$ ; however, now the signals are sparse in the discrete cosine transform (DCT) domain. We consider four reconstruction experiments. First, we quantize the Nyquist-rate signal with a bit-depth of  $\beta$  bits per time-domain sample (and optimal quantizer scale) and perform linear reconstruction (i.e., we just use the quantized samples as sample values). Second, we apply BPDN to the quantized Nyquist-rate samples with optimal choice of noise parameter, thus denoising the signal using a sparsity model. Third, we draw a new Gaussian matrix with  $M = N$ , quantize the measurements to  $\beta$  bits, again at optimal quantizer scale, and reconstruct using BPDN. Fourth, we draw a new Gaussian matrix with  $M = \beta N$  and compute measurements, quantize to one bit per measurement by maintaining their sign, and perform reconstruction with BIHT. Note that the same total number of bits is used in each experiment.

Figure 7 depicts the average SNR obtained by performing 100 of the above trials. The linear, BPDN, Gaussian measurements with BPDN, and BIHT reconstructions are depicted by solid, dashed, dash-circled, and dash-dotted lines, respectively. The linear reconstruction has a slope of 6.02dB/bit-depth, exhibiting a well-known trade-off for conventional uniform quantization. The BPDN reconstruction (without projections) follows the same trend, but obtains an SNR that is at least 10dB higher than the linear reconstruction. This is because BPDN imposes the sparse signal model to denoise the signal. We see about the same performance with the Gaussian projections at  $M = N$ , although it performs slightly worse than without projections since the Gaussian measurements require a slightly larger quantizer range. Similarly to the results in Fig. 6, in low Nyquist bit-depth regimes ( $\beta < 6$ ), 1-bit CS achieves a significantly higher SNR than the other two techniques. When  $6 < \beta < 8$ , 1-bit CS is competitive with the BPDN scenario. Thus, for a fixed number of bits, 1-bit CS is competitive to conventional sampling with uniform quantization, especially in low bit-depth regimes.

## VI. DISCUSSION

In this paper we have developed a rigorous mathematical foundation for 1-bit CS. Specifically, we have demonstrated a lower bound on reconstruction error as a function of the number of measurements and the sparsity of the signal. We have demonstrated that Gaussian random projections almost reach this lower bound (up to a log factor) in the noiseless case. This behavior is consistent with and extends existing results in the literature on multibit scalar quantization and 1-bit quantization of non-sparse signals.

We have also introduced reconstruction robustness guarantees through the binary  $\epsilon$ -stable embedding (BeSE) property. This property can be thought of as extending the RIP to 1-bit quantized measurements. To our knowledge, this is the first time such a property has been introduced in the context of quantization. To be able to use this property we showed that random constructions using Gaussian pdf (or more generally using isotropic pdf in the signal space  $\mathbb{R}^N$ ) generate such embeddings with high probability. This construction class is still very limited compared to the numerous random constructions known for generating RIP matrices. Extending this class with other constructions is an interesting topic for future research.

Using the BeSE, we have proven that 1-bit CS systems are robust to measurement noise added before quantiza-

tion as well as to signals that are not exactly sparse but compressible.

We have introduced a new 1-bit CS algorithm, BIHT, that achieves better performance over previous algorithms in the noiseless case. This improvement is due to the enforcement of consistency using a one-sided linear objective, as opposed to a quadratic one. The linear objective is similar to the hinge loss from the machine learning literature.

We remind the reader that the central goal of this paper has been signal *acquisition* with quantization. As explained previously, one motivation for our work is the development of very high speed samplers. In this case, we are interested in building fast samplers by relaxing the requirements on the primary hardware burden, the quantizer. Such devices are susceptible to noise. Thus, while our noiseless results extend previous 1-bit quantization results (e.g., see [51] and [49]) to the sparse signal model setting and are of theoretical interest, a major contribution has been the further development of the robust guarantees, even if they produce error rates that seem suboptimal when compared to the noiseless case.

A number of interesting questions remain unanswered. As we discuss in Section III-B earlier, we have found that the BeSE holds for Gaussian matrices with angular error decay roughly on the order of  $O(\sqrt{K/M})$  worse than optimal. One question is whether this gap can be closed with an alternative derivation, or whether it is a fundamental requirement for stability. Another useful pursuit would be to provide a more rigorous understanding of the discrepancy between the performance of the one-sided  $\ell_1$  and  $\ell_2$  objectives. Analysis of the performance behavior might lead to better one-sided functions.

## ACKNOWLEDGMENTS

Thanks to Rachel Ward for pointing us to the right reference with regards to the lower bound (16) used in Appendix A and recommending several useful articles as well as Vivek Goyal for pointing us to additional prior work in this area. Thanks to Zaiwen Wen and Wotao Yin for sharing some of the data from [30] for our algorithm comparisons, as well as engaging in numerous conversations on this topic. Thanks also to Nathan Srebro for his advice and discussion related to the one-sided penalty comparison and connections to binary classification. Finally, thanks to Yaniv Plan and anonymous reviewers for their useful advices and remarks for improving this paper, and in particular the proof of Theorem 1 with respect to the optimality of the



$\Sigma_K^*$  covering.

## APPENDIX A

### LEMMA 1: INTERSECTIONS OF ORTHANTS BY SUBSPACES

While there are  $2^M$  available quantization points provided by 1-bit measurements, a  $K$ -sparse signal will not use all of them. To understand how effectively the quantization bits are used, we first investigate how the  $K$ -dimensional subspaces projected from the  $N$ -dimensional  $K$ -sparse signal spaces intersect orthants in the  $M$ -dimensional measurement space, as shown in Fig 1 for  $K = 2$  and  $M = 3$ .

We use  $I(M, K)$  to denote the maximum number of orthants in  $M$  dimensions intersected by a  $K$ -dimensional subspaces. A bound of for  $I(M, K)$  is developed in [66, 67]:

$$I(M, K) \leq 2 \sum_{l=0}^{K-1} \binom{M-1}{l}. \quad (16)$$

For  $K \leq M/2$ , this simplifies to  $I(M, K) \leq 2K \binom{M-1}{K-1}$ .

Using  $\binom{p}{q-1} + \binom{p}{q} = \binom{p+1}{q}$  we can also derive a simple bound on (16) for  $K \leq M$ . We observe that

$$\begin{aligned} \sum_{l=0}^{K-1} \binom{M-1}{l} &= \binom{M-1}{1} + \sum_{l=2}^{K-1} \binom{M-1}{l} \\ &\leq \binom{M}{1} + \sum_{l=2}^{K-1} \binom{M}{l} \leq \sum_{l=1}^{K-1} \binom{M}{l}. \end{aligned}$$

Repeating the same argument, we find  $\sum_{l=1}^{K-1} \binom{M}{l} \leq \sum_{l=2}^{K-1} \binom{M+1}{l} \leq \dots \leq \binom{M+K-2}{K-1}$  and finally

$$I(M, K) \leq 2 \binom{M+K-2}{K-1} \leq 2 \binom{2M-1}{K-1} \quad (17)$$

$$= 2 \left(\frac{K}{2M}\right) \binom{2M}{K} \leq 2^K \left(\frac{eM}{K}\right)^K, \quad (18)$$

using the bound  $\binom{M}{K} \leq \left(\frac{eM}{K}\right)^K$ .

## APPENDIX B

### THEOREM 1: DISTRIBUTING SIGNALS TO QUANTIZATION POINTS

To prove Theorem 1 we consider how the available quantization points optimally cover the set of signals  $\Sigma_K^* = \{\mathbf{x} \in \mathbb{R}^N : \|\mathbf{x}\| = 1, \|\mathbf{x}\|_0 \leq K\}$ . This set corresponds to the union of  $L := \binom{N}{K}$   $K$ -dimensional unit spheres  $\bigcup_{i \in [L]} S_i$ , each  $S_i = \{\mathbf{x} \in \mathbb{R}^N : \|\mathbf{x}\| = 1, \text{supp } \mathbf{x} \subset T_i\}$  being associated to one support  $T_i \subset [N]$  taken amongst the  $L$  available  $K$ -length supports of  $\mathbb{R}^N$ . The cover should be optimal with respect to the worst case distance, denoted by  $r$ , of any point in  $\Sigma_K^*$  to its closest quantization point. Our goal is to determine a lower bound on the best-case  $r$  we can achieve.

Unfortunately, determining the optimal cover of  $\Sigma_K^*$  is not straightforward. For instance, optimally covering each  $S_i$  individually does not produce an optimal cover for their union. Indeed, at the intersection of different  $K$ -dimensional spheres in  $\Sigma_K^*$  there are  $K$ -sparse signals, with different support, very close to each other. A single quantization point in  $\mathbb{R}^N$  could be close to all those signals, whereas independent covers of each  $S_i$  would have to use a different quantization point to represent the signals on each sphere in that intersection.

Thus, instead of determining the optimal cover of  $\Sigma_K^*$ , we establish a lower bound on  $r$  required to cover a subset  $\tilde{\Sigma}_K^*$  of  $\Sigma_K^*$  using the same number of points. A cover of  $\Sigma_K^*$  with a smaller  $r$  would not be possible, since that would also cover  $\tilde{\Sigma}_K^*$  with the same or smaller  $r$ . Therefore, this  $r$  establishes a lower bound for the cover of  $\Sigma_K^*$ . To establish the bound, we pick  $\tilde{\Sigma}_K^*$  such that the neighborhood around the intersection of the balls is not included. Specifically, we pick

$$\tilde{\Sigma}_K^* := \bigcup_i \tilde{S}_i \subset \Sigma_K^*,$$

with  $\tilde{S}_i := \{\mathbf{x} \in S_i : \forall k \in T_i, |x_k| > \sqrt{2}r\} \subset S_i$ .

In other words, we pick the subset of each  $K$ -ball, such that all the nonzero coordinates of the signals in the subset are greater than  $\sqrt{2}r$ . The union of those subsets for all possible supports comprises  $\tilde{\Sigma}_K^*$ . This choice ensures that any signal  $\mathbf{x} \in \tilde{S}_i$  has distance at least  $2r$  from any signal in any other  $\tilde{S}_j, j \neq i$ , and therefore both cannot be close to a common quantization point in  $\mathbb{R}^N$  with distance  $r$  from each. Notice that  $\tilde{S}_i$  is non empty as soon as  $r < 1/(\sqrt{2}K)$ .

With this choice of  $\tilde{\Sigma}_K^*$ , an optimal covering can be obtained by merging the optimal coverings of each  $\tilde{S}_i$  for  $i \in [L]$ . For an optimal covering of distance  $r$ , each of the elements in  $\tilde{S}_i$  should belong to some ball of radius  $r$  centered at the quantization point. Thus, each quantization point and its corresponding  $r$ -ball should cover as large an area of  $\tilde{S}_i$  as possible. This is achieved when the quantization point is on  $\tilde{S}_i$ , and the intersection of the ball with  $\tilde{S}_i$  is a spherical cap of radius  $r$  in a  $K$ -dimensional subspace [68]. Thus, for the spherical caps to cover  $\tilde{S}_i$ , the total area of all spherical caps in the cover should be greater than the area of  $\tilde{S}_i$ . Furthermore, since the overall cover of  $\tilde{\Sigma}_K^*$  is composed of separate cover of each  $\tilde{S}_i$ , the total area of all spherical caps used for the cover should be greater than the area of  $\tilde{\Sigma}_K^*$ .

Therefore, since  $2^K L \binom{M}{K}$  quantization points and corresponding spherical caps are available, according to Lemma 1 and Corollary 1, to cover  $L$  subsets of  $K$ -

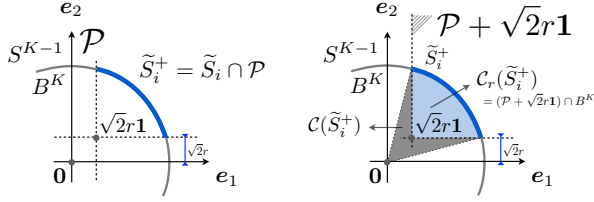


Fig. 8: The geometry of our problem in  $\mathbb{R}^2$ . We only need to consider one orthant,  $\mathcal{P}$ , as the problem is the same for all orthants. To measure the surface area of  $\tilde{S}_i^+$  relative to the surface of  $S^{K-1}$ , we consider the ratio of the volume of  $\mathcal{C}(\tilde{S}_i^+)$  with respect to this of  $B^K$ . We lower bound the former using  $\mathcal{C}_r(\tilde{S}_i^+) \subset \mathcal{C}(\tilde{S}_i^+)$ .

spheres, as described above, the cover should satisfy

$$2^K L \binom{M}{K} \kappa(r) \geq \sigma(\tilde{\Sigma}_K^*), \quad (19)$$

where  $\sigma(\cdot)$  denotes the rotationally invariant area measure of the  $K$ -sphere  $S_i$  and  $\kappa(r)$  denotes the surface area of a spherical cap of radius  $r$  measured using  $\sigma$ .

To determine the smallest  $r$  satisfying (19), we thus need to measure the set  $\tilde{\Sigma}_K^*$ . Choosing one  $i \in [L]$  we have  $\sigma(\tilde{\Sigma}_K^*) = L\sigma(\tilde{S}_i)$  since the sets  $\tilde{S}_k$  ( $k \in [L]$ ) are disjoint with identical area. We first show that

$$\sigma(\tilde{S}_i) \geq (1 - r\sqrt{2K})^K \sigma(S^{K-1}), \quad (20)$$

where  $S^{K-1}$  is a  $K$ -dimensional sphere in  $\mathbb{R}^K$ . Note that this bound is tight at two ends: at  $r = 0$  where  $\tilde{S}_i = S_i$  (almost everywhere) and at  $r = 1/\sqrt{2K}$  for which  $\tilde{S}_i = \emptyset$ .

To prove (20) we assume without loss of generality  $T_i = \{1, \dots, K\}$  and consider  $S_i = S^{K-1} = \{\mathbf{x} \in \mathbb{R}^K : \|\mathbf{x}\| = 1\}$  and  $\tilde{S}_i = \{\mathbf{x} \in \mathbb{R}^K : \|\mathbf{x}\| = 1, \min |x_k| > \sqrt{2}r\}$ . We define the intersection  $\tilde{S}_i^+ := \tilde{S}_i \cap \mathcal{P}$  with the positive orthant  $\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^K : \forall i \in [K], x_i \geq 0\}$ . By symmetry,  $\sigma(\tilde{S}_i) = 2^K \sigma(\tilde{S}_i^+)$  since there are  $2^K$  orthants in  $\mathbb{R}^K$ . As described in [68], the ratio between the measure of  $\tilde{S}_i^+$  and the one of the full sphere  $S^{K-1}$  equals the ratio between the volume occupied by a cone formed by  $\tilde{S}_i^+$  in the unit ball  $B^K \subset \mathbb{R}^K$  and the volume of this ball, i.e.,

$$\frac{\sigma(\tilde{S}_i^+)}{\sigma(S^{K-1})} = \frac{\mu(\mathcal{C}(\tilde{S}_i^+))}{\mu(B^K)}, \quad (21)$$

where  $\mu$  is the Lebesgue measure in  $\mathbb{R}^K$  and  $\mathcal{C}(A) = \{t\mathbf{a} : t \in [0, 1], \mathbf{a} \in A\} \subset B^K$  is the portion of the cone (with apex on the origin and restricted to  $B^K$ ) formed by the subset  $A \subset S^{K-1}$ . The geometry of our problem is made clear in Figure 8.

To measure the volume of  $\mathcal{C}(\tilde{S}_i^+)$ , writing  $\mathbf{1} = (1, \dots, 1)^T \in \mathbb{R}^K$ , we first define the set

$$\mathcal{C}_r(\tilde{S}_i^+) := (\mathcal{P} + \sqrt{2}r\mathbf{1}) \cap B^K,$$

also shown in the figure, which is non-empty for  $\sqrt{2}r \leq 1/\sqrt{K}$ . Since  $\mathcal{C}_r(\tilde{S}_i^+) \cap S_i = \tilde{S}_i^+$  and  $\sqrt{2}r\mathbf{1} \in \mathcal{C}(\tilde{S}_i^+)$ , it is straightforward to show that  $\mathcal{C}_r(\tilde{S}_i^+) \subset \mathcal{C}(\tilde{S}_i^+)$ , and therefore the measure of  $\mathcal{C}_r(\tilde{S}_i^+)$  is a lower bound to the measure of  $\mathcal{C}(\tilde{S}_i^+)$ . Furthermore, by the translation invariance of  $\mu$ ,

$$\begin{aligned} \mu(\mathcal{C}_r(\tilde{S}_i^+)) &= \mu((\mathcal{P} + \sqrt{2}r\mathbf{1}) \cap B^K) \\ &= \mu(\mathcal{P} \cap (B^K - \sqrt{2}r\mathbf{1})), \end{aligned}$$

where  $B^K - \sqrt{2}r\mathbf{1}$  is the unit ball centered on  $-\sqrt{2}r\mathbf{1}$ . Setting  $\alpha = (1 - r\sqrt{2K})$  it follows that  $\|\mathbf{u} + \sqrt{2}r\mathbf{1}\| \leq 1$  for any  $\mathbf{u} \in \mathbb{R}^K$ ,  $\|\mathbf{u}\| \leq \alpha$ . Consequently,  $\alpha B^K \subset B^K - \sqrt{2}r\mathbf{1}$  and  $\mathcal{P} \cap \alpha B^K \subset \mathcal{P} \cap (B^K - \sqrt{2}r\mathbf{1})$ . Therefore, the measure of the positive orthant of a  $K$ -ball with radius  $\alpha$  lower bounds the measure of  $\mathcal{C}(\tilde{S}_i^+)$ .

Putting everything together we obtain

$$\mu(\mathcal{C}(\tilde{S}_i^+)) \geq \mu(\mathcal{P} \cap \alpha B^K) = \alpha^K 2^{-K} \mu(B^K),$$

which, using (21), implies that  $\sigma(\tilde{S}_i^+)/\sigma(S^{K-1}) \geq \alpha^K 2^{-K}$  and

$$\sigma(\tilde{\Sigma}_K^*) = L\sigma(\tilde{S}_i) \geq (1 - r\sqrt{2K})^K L\sigma(S^{K-1}).$$

In turn, (19) becomes

$$2^K L \binom{M}{K} \kappa(r) \geq (1 - r\sqrt{2K})^K L\sigma(S^{K-1}),$$

where  $\kappa(r) \leq r^K \sigma(S^{K-1})$  [68]. From  $\binom{M}{K} \leq (eM/K)^K$ , the result follows:

$$\begin{aligned} 2^K r^K \left(\frac{eM}{K}\right)^K &\geq (1 - r\sqrt{2K})^K \\ \Rightarrow r &\geq \left(\frac{2eM}{K} + \sqrt{2K}\right)^{-1} = K/(2eM + \sqrt{2}K^{3/2}). \end{aligned}$$

## APPENDIX C

### THEOREM 2: OPTIMAL PERFORMANCE VIA GAUSSIAN PROJECTIONS

To prove Theorem 2, we follow the procedure given in [46, Theorem 3.3]. We begin by restricting our analysis to the support set  $T \subset [N] := \{1, \dots, N\}$  with  $|T| \leq D \leq N$ , and thus we consider vectors that lie on the (sub) sphere  $\Sigma^*(T) = \{\mathbf{x} : \text{supp } \mathbf{x} \subset T, \|\mathbf{x}\|_2 = 1\} \subset \mathbb{R}^N$ . We remind the reader that  $B_r(\mathbf{x}) := \{\mathbf{a} \in S^{N-1} : \|\mathbf{x} - \mathbf{a}\|_2 < r\}$  is the ball of unit norm vectors of Euclidean distance  $r > 0$  around  $\mathbf{x}$ , and we write  $B_r^*(\mathbf{x}) = B_r(\mathbf{x}) \cap \Sigma^*(T)$  as in Section III-B.

Let us fix a radius  $\delta > 0$  to be precised later. The sphere  $\Sigma^*(T)$  can be covered with a finite set  $Q_\delta \subset \Sigma^*(T)$  of no more than  $(3/\delta)^D$  points such that, for any  $\mathbf{w} \in \Sigma^*(T)$ , there exists a  $\mathbf{q} \in Q_\delta$  with  $\mathbf{w} \in B_\delta^*(\mathbf{q})$  [16].

Using the notation  $d_S$  defined in Sec. III-A, given a vector  $\boldsymbol{\varphi} \sim \mathcal{N}^{N \times 1}(0, 1)$  and two distinct points  $\mathbf{p}$  and  $\mathbf{q}$  in  $Q_\delta$ , we have that

$$\begin{aligned} \mathbb{P}[\forall \mathbf{u} \in B_\delta^*(\mathbf{p}), \forall \mathbf{v} \in B_\delta^*(\mathbf{q}) : \text{sign } \boldsymbol{\varphi}^T \mathbf{u} \neq \text{sign } \boldsymbol{\varphi}^T \mathbf{v}] \\ \geq d_S(\mathbf{p}, \mathbf{q}) - \sqrt{\frac{\pi}{2} D} \delta, \end{aligned}$$

from Lemma 6 (given in Appendix D). Since for all  $\mathbf{u} \in B_\delta^*(\mathbf{p})$  and  $\mathbf{v} \in B_\delta^*(\mathbf{q})$

$$\begin{aligned} \pi d_S(\mathbf{p}, \mathbf{q}) &\geq 2 \sin(\frac{\pi}{2} d_S(\mathbf{p}, \mathbf{q})) = \|\mathbf{p} - \mathbf{q}\|_2 \\ &\geq \|\mathbf{u} - \mathbf{v}\|_2 - 2\delta, \end{aligned}$$

we can write for any  $\epsilon_o > 0$

$$\begin{aligned} \mathbb{P}[\forall \mathbf{u} \in B_\delta^*(\mathbf{p}), \forall \mathbf{v} \in B_\delta^*(\mathbf{q}) : \\ \text{sign } \boldsymbol{\varphi}^T \mathbf{u} \neq \text{sign } \boldsymbol{\varphi}^T \mathbf{v} \mid \|\mathbf{u} - \mathbf{v}\|_2 > \epsilon_o] \\ \geq \frac{\epsilon_o}{\pi} - (\frac{2}{\pi} + \sqrt{\frac{\pi}{2} D}) \delta. \end{aligned}$$

By setting  $\delta = \pi \epsilon_o / (4 + \pi \sqrt{2\pi D})$  (and reversing the inequality), we obtain

$$\begin{aligned} \mathbb{P}[\exists \mathbf{u} \in B_\delta^*(\mathbf{p}), \exists \mathbf{v} \in B_\delta^*(\mathbf{q}) : \\ \text{sign } (\boldsymbol{\varphi}^T \mathbf{u}) = \text{sign } (\boldsymbol{\varphi}^T \mathbf{v}) \mid \|\mathbf{u} - \mathbf{v}\|_2 > \epsilon_o] \leq 1 - \frac{\epsilon_o}{2}. \end{aligned}$$

Thus, for  $M$  different random vectors  $\boldsymbol{\varphi}_i$  arranged in  $\Phi = (\boldsymbol{\varphi}_1, \dots, \boldsymbol{\varphi}_M)^T \sim \mathcal{N}^{M \times N}(0, 1)$ , and for the associated mapping  $A$  defined in (3), we get

$$\begin{aligned} \mathbb{P}[\exists \mathbf{u} \in B_\delta^*(\mathbf{p}), \exists \mathbf{v} \in B_\delta^*(\mathbf{q}) : \\ A(\mathbf{u}) = A(\mathbf{v}) \mid \|\mathbf{u} - \mathbf{v}\|_2 > \epsilon_o] \leq (1 - \frac{\epsilon_o}{2})^M. \end{aligned}$$

In other words, we have found a bound on the probability that two vectors' measurements are consistent, even if their Euclidean distance is greater than  $\epsilon_o$ , but only for vectors in the restricted (sub) sphere  $\Sigma^*(T)$ . Now we seek to cover the rest of the space  $\Sigma_K^*$  (unit norm  $K$ -sparse signals).

Since there are no more than  $\binom{|Q_\delta|}{2} \leq (|Q_\delta|)^2 \leq (3/\delta)^{2D}$  pairs of distinct points in  $Q_\delta$ , we find

$$\begin{aligned} \mathbb{P}[\exists \mathbf{u}, \mathbf{v} \in \Sigma^*(T) : A(\mathbf{u}) = A(\mathbf{v}) \mid \|\mathbf{u} - \mathbf{v}\|_2 > \epsilon_o] \\ \leq \left(\frac{1}{\pi \epsilon_o} (12 + 3\pi \sqrt{2\pi D})\right)^{2D} (1 - \frac{\epsilon_o}{2})^M. \end{aligned}$$

To obtain the final bound, we observe that any pair of unit  $K$ -sparse vectors  $\mathbf{x}$  and  $\mathbf{s}$  in  $\Sigma_K^*$  belongs to some  $\Sigma^*(T)$  with  $T = \text{supp } \mathbf{x} \cup \text{supp } \mathbf{s}$  and  $|T| \leq 2K$ . There

are no more than  $\binom{N}{2K} \leq (eN/2K)^{2K}$  of such sets  $T$ , and thus setting  $D = 2K$  above yields

$$\begin{aligned} \mathbb{P}[\exists \mathbf{u}, \mathbf{v} \in \Sigma_K^* : A(\mathbf{u}) = A(\mathbf{v}) \mid \|\mathbf{u} - \mathbf{v}\|_2 > \epsilon_o] \\ \leq \left(\frac{eN}{2K}\right)^{2K} \left(\frac{1}{\pi \epsilon_o} (12 + 6\pi \sqrt{\pi K})\right)^{4K} (1 - \frac{\epsilon_o}{2})^M \\ \leq e^{2K \log(\frac{eN}{2K}) + 4K \log(\frac{1}{\pi \epsilon_o} (12 + 6\pi \sqrt{\pi K})) - M \frac{\epsilon_o}{2}}, \end{aligned}$$

where the second inequality follows from  $1 - \frac{\epsilon_o}{2} \leq \exp \frac{\epsilon_o}{2}$ . By upper bounding this probability by  $\eta$  and solving for  $M$ , we obtain

$$M \geq \frac{2}{\epsilon_o} \left(2K \log \frac{eN}{2K} + 4K \log \left(\frac{12 + 6\pi \sqrt{\pi K}}{\pi \epsilon_o}\right) + \log \frac{1}{\eta}\right).$$

Since  $K \geq 1$ , we have that  $\frac{1}{\pi} (12 + 6\pi \sqrt{\pi K}) < 17 \sqrt{\frac{2K}{e}}$ , and thus the previous relation is then satisfied when

$$\begin{aligned} M &\geq \frac{2}{\epsilon_o} \left(2K \log \frac{eN}{2K} + 4K \log \left(\frac{17\sqrt{2}}{\sqrt{e} \epsilon_o} \sqrt{K}\right) + \log \frac{1}{\eta}\right) \\ &= \frac{2}{\epsilon_o} \left(2K \log N + 4K \log \left(\frac{17}{\epsilon_o}\right) + \log \frac{1}{\eta}\right). \end{aligned}$$

## APPENDIX D

### LEMMA 3: CONCENTRATION OF MEASURE FOR $\delta$ -BALLS

Since  $T \subset [N]$  is fixed with size  $|T| = D$ , proving Lemma 3 amounts to showing that, for any fixed  $\epsilon > 0$  and  $0 \leq \delta \leq 1$ , given a Gaussian matrix  $\Phi \in \mathbb{R}^{M \times D}$ , the mapping  $A : \mathbb{R}^D \rightarrow \mathcal{B}^M$  defined as  $A(\mathbf{u}) = \text{sign}(\Phi \mathbf{u})$ , and for any fixed  $\mathbf{x}, \mathbf{s} \in S^{D-1}$ , we have

$$\begin{aligned} \mathbb{P}\left(\forall \mathbf{u} \in B_\delta^*(\mathbf{x}), \forall \mathbf{v} \in B_\delta^*(\mathbf{s}), \right. \\ \left. |d_H(A(\mathbf{u}), A(\mathbf{v})) - d_S(\mathbf{x}, \mathbf{s})| \leq \epsilon + \sqrt{\frac{\pi}{2} D} \delta\right) \\ \geq 1 - 2e^{-2\epsilon^2 M}, \end{aligned}$$

where, in this case,  $B_\delta^*(\mathbf{p}) = (B_\delta(\mathbf{p}) \cap S^{D-1}) \subset \mathbb{R}^D$  for any  $\mathbf{p} \in \mathbb{R}^D$ .

Given  $\mathbf{u}' \in B_\delta^*(\mathbf{x})$  and  $\mathbf{v}' \in B_\delta^*(\mathbf{s})$ , the quantity  $M d_H(A(\mathbf{u}'), A(\mathbf{v}'))$  is the sum  $\sum_i A_i(\mathbf{u}') \oplus A_i(\mathbf{v}')$ , where  $A_i(\mathbf{u}')$  stands for the  $i^{\text{th}}$  component of  $A(\mathbf{u}')$ . For one index  $1 \leq i \leq M$ ,

$$\begin{aligned} Z_i^+ &:= \max \{A_i(\mathbf{u}) \oplus A_i(\mathbf{v}) : \mathbf{u} \in B_\delta^*(\mathbf{x}), \mathbf{v} \in B_\delta^*(\mathbf{s})\} \\ &\geq A_i(\mathbf{u}') \oplus A_i(\mathbf{v}'), \\ Z_i^- &:= \min \{A_i(\mathbf{u}) \oplus A_i(\mathbf{v}) : \mathbf{u} \in B_\delta^*(\mathbf{x}), \mathbf{v} \in B_\delta^*(\mathbf{s})\} \\ &\leq A_i(\mathbf{u}') \oplus A_i(\mathbf{v}'), \end{aligned}$$

and therefore

$$\begin{aligned} Z^- &:= \sum_{i=1}^M Z_i^- \leq M d_H(A(\mathbf{u}'), A(\mathbf{v}')) \\ &\leq \sum_{i=1}^M Z_i^+ =: Z^+. \end{aligned}$$

Of course, the occurrence of  $Z_i^+ = 0$  ( $Z_i^- = 1$ ) means that all vector pairs taken separately in  $B_\delta^*(\mathbf{x})$  and  $B_\delta^*(\mathbf{s})$  have consistent (or respectively, inconsistent) measurements on the  $i^{\text{th}}$  sensing component  $A_i$ . More precisely, since  $\varphi_i \sim \mathcal{N}^{N \times 1}(0, 1)$ ,  $Z_i^\pm$  are binary random variables such that  $\mathbb{P}[Z_i^+ = 1] = 1 - p_0$  and  $\mathbb{P}[Z_i^- = 1] = p_1$  independently of  $i$ , where the probabilities  $p_0$  and  $p_1$  are defined by

$$\begin{aligned} p_0(\mathbf{x}, \mathbf{s}, \delta) &= \mathbb{P}[Z_i^+ = 0] \\ &= \mathbb{P}[\forall \mathbf{u} \in B_\delta^*(\mathbf{x}), \forall \mathbf{v} \in B_\delta^*(\mathbf{s}), A_i(\mathbf{u}) = A_i(\mathbf{v})], \\ p_1(\mathbf{x}, \mathbf{s}, \delta) &= \mathbb{P}[\forall \mathbf{u} \in B_\delta^*(\mathbf{x}), \forall \mathbf{v} \in B_\delta^*(\mathbf{s}), A_i(\mathbf{u}) \neq A_i(\mathbf{v})]. \end{aligned}$$

In summary,  $Z^+$  and  $Z^-$  are binomially distributed with  $M$  trials and probability of success  $1 - p_0$  and  $p_1$ , respectively. Furthermore, we have that  $\mathbb{E}Z^+ = M(1 - p_0)$  and  $\mathbb{E}Z^- = Mp_1$ , thus by the Chernoff-Hoeffding inequality,

$$\begin{aligned} \mathbb{P}[Z^+ > M(1 - p_0) + M\epsilon] &\leq e^{-2M\epsilon^2}, \\ \mathbb{P}[Z^- < Mp_1 - M\epsilon] &\leq e^{-2M\epsilon^2}. \end{aligned}$$

This indicates that with a probability higher than  $1 - 2e^{-2M\epsilon^2}$ , we have

$$p_1 - \epsilon \leq d_H(A(\mathbf{u}'), A(\mathbf{v}')) \leq (1 - p_0) + \epsilon.$$

The final result follows by lower bounding  $p_0$  and  $p_1$  as in Lemma 6.

**Lemma 6.** *Given  $0 \leq \delta < 1$  and two unit vectors  $\mathbf{x}, \mathbf{s} \in S^{D-1}$ , we have*

$$\begin{aligned} p_0 &= \mathbb{P}[\forall \mathbf{u} \in B_\delta^*(\mathbf{x}), \forall \mathbf{v} \in B_\delta^*(\mathbf{s}), \\ &\quad \text{sign} \langle \varphi, \mathbf{u} \rangle = \text{sign} \langle \varphi, \mathbf{v} \rangle] \\ &\geq 1 - d_S(\mathbf{x}, \mathbf{s}) - \sqrt{\frac{\pi}{2}D} \delta, \quad (22) \end{aligned}$$

$$\begin{aligned} p_1 &= \mathbb{P}[\forall \mathbf{u} \in B_\delta^*(\mathbf{x}), \forall \mathbf{v} \in B_\delta^*(\mathbf{s}), \\ &\quad \text{sign} \langle \varphi, \mathbf{u} \rangle \neq \text{sign} \langle \varphi, \mathbf{v} \rangle] \\ &\geq d_S(\mathbf{x}, \mathbf{s}) - \sqrt{\frac{\pi}{2}D} \delta. \quad (23) \end{aligned}$$

*Proof of Lemma 6:* We begin by introducing some useful properties of Gaussian vector distribution. If  $\varphi \sim \mathcal{N}^{D \times 1}(0, 1)$ , the probability that  $\varphi \in \mathcal{A} \subset \mathbb{R}^D$  is simply the measure  $\mu$  of  $\mathcal{A}$  with respect to the standard Gaussian density  $\gamma(\varphi) = \frac{1}{(2\pi)^{D/2}} e^{-\|\varphi\|^2/2}$ , i.e.,

$$\mathbb{P}[\varphi \in \mathcal{A}] = \mu(\mathcal{A}) = \int_{\mathcal{A}} d^D \varphi \gamma(\varphi),$$

with  $\mu(\mathbb{R}^D) = 1$ . It may be easier to perform this integration over a hyper-spherical set of coordinates

measured in a basis defined by the vectors  $\mathbf{x}$  and  $\mathbf{s}$ . This is possible since the pdf  $\gamma$  is rotationally invariant.

Specifically, we consider the canonical basis  $\mathcal{E} = \{e_1, \dots, e_D\}$  of  $\mathbb{R}^D$  where, by using the cross product  $\wedge$  in  $\mathbb{R}^D$ ,  $e_1 := (\mathbf{x} \wedge \mathbf{s}) / \|\mathbf{x} \wedge \mathbf{s}\|_2$ ,  $e_D := \mathbf{x}$  and  $e_{D-1} := e_D \wedge e_1$ , while the other vectors  $\{e_k : 2 \leq k \leq D-2\}$  are defined arbitrarily for completing the basis. In this system, the “ $\mathbf{x}\mathbf{s}$ ” plane is equivalent to the plane spanned by  $e_D$  and  $e_{D-1}$ . Moreover, any vector  $\varphi \in \mathbb{R}^D$  can be represented by the spherical coordinates  $(r, \phi_1, \dots, \phi_{D-1})$  where  $r = \|\varphi\|_2 \in \mathbb{R}_+$ ,  $(\phi_1, \dots, \phi_{D-2}) \in [0, \pi]^{D-2}$  corresponds to the vector angles in each dimension, and  $\phi_{D-1} \in [0, 2\pi]$  being the angle formed by the projection of  $\varphi$  in the “ $\mathbf{x}\mathbf{s}$ ” plane with  $\mathbf{x} = e_D$ .

The change of coordinates between the Cartesian and the spherical representations of  $\varphi$  in  $\mathcal{E}$  is then defined as  $\varphi_1 = r \cos \phi_1$ ,  $\varphi_2 = r \sin \phi_1 \cos \phi_2$ , ...,  $\varphi_{D-1} = r \sin \phi_1 \cdots \sin \phi_{D-2} \cos \phi_{D-1}$ , and  $\varphi_D = r \sin \phi_1 \cdots \sin \phi_{D-2} \sin \phi_{D-1}$ , while, conversely,  $r = \|\varphi\|_2$ ,  $\tan \phi_1 = (\varphi_D^2 + \cdots + \varphi_2^2)^{1/2} / \varphi_1$ , ...,  $\tan \phi_{D-2} = (\varphi_D^2 + \varphi_{D-1}^2)^{1/2} / \varphi_{D-2}$ , and  $\tan \phi_{D-1} = \varphi_D / \varphi_{D-1}$ .<sup>9</sup>

We now seek a lower bound on  $p_1$ . Computing this probability amounts to estimating

$$\begin{aligned} p_1 &= \mathbb{P}[\forall \mathbf{u} \in B_\delta^*(\mathbf{x}), \forall \mathbf{v} \in B_\delta^*(\mathbf{s}), \langle \varphi, \mathbf{u} \rangle \langle \varphi, \mathbf{v} \rangle \leq 0] \\ &= \mu(\mathcal{W}_\delta), \quad (24) \end{aligned}$$

where  $\mathcal{W}_\delta := \{\varphi : \langle \varphi, \mathbf{u} \rangle \langle \varphi, \mathbf{v} \rangle \leq 0, \forall \mathbf{u} \in B_\delta^*(\mathbf{x}), \forall \mathbf{v} \in B_\delta^*(\mathbf{s})\}$  is the set of all vectors  $\varphi$  such that its inner product with  $\mathbf{u}$  and  $\mathbf{v}$  result in different signs.

Note that if  $B_\delta^*(\mathbf{x}) \cap B_\delta^*(\mathbf{s}) \neq \emptyset$ , then  $p_1 = 0$  since  $p_1 \leq \mathbb{P}[\forall \mathbf{u} \in B_\delta^*(\mathbf{x}) \cap B_\delta^*(\mathbf{s}), \langle \varphi, \mathbf{u} \rangle^2 = 0] = 0$ . This non-empty intersection is avoided when  $d_S(\mathbf{x}, \mathbf{s}) \geq \frac{4}{\pi} \arcsin \delta / 2$ . Furthermore, since  $\arcsin \lambda \leq \frac{\pi}{2} \lambda$  for any  $0 \leq \lambda \leq 1$ , this occurs if  $d_S(\mathbf{x}, \mathbf{s}) \geq \delta$ .

The remainder of the proof is devoted to finding an appropriate way to integrate the set  $\mathcal{W}_\delta$ . To this end, we begin by demonstrating that estimating  $p_1$  can be simplified with the following equivalence (proved just after the completion of the proof of Lemma 6).

<sup>9</sup>This change of coordinates can be very convenient. For instance, the proof of Lemma 2 relies on the computation  $\mathbb{P}[A_i(\mathbf{x}) \neq A_i(\mathbf{s})] = \mu(\mathcal{A} = \{\varphi : \phi_{D-1} \in [0, \pi d_S(\mathbf{x}, \mathbf{s})] \cup [\pi, \pi + \pi d_S(\mathbf{x}, \mathbf{s})]\}) = d_S(\mathbf{x}, \mathbf{s})$ , since for (almost) all  $\varphi \in \mathcal{A}$ ,  $\mathbf{x}$  and  $\mathbf{s}$  live in the two different subvolumes determined by the plane  $\{\mathbf{u} : \langle \varphi, \mathbf{u} \rangle = 0\}$  [55, 56].

**Lemma 7.** *The set  $\mathcal{W}_\delta \subset \mathbb{R}^D$  is equal to the set*

$$\mathcal{V}_\delta^- = \{\varphi : \langle \varphi, \mathbf{x} \rangle \langle \varphi, \mathbf{s} \rangle \leq 0, \\ \|\mathbf{x} - \mathcal{P}_{\Pi(\varphi)} \mathbf{x}\| \geq \delta, \|\mathbf{s} - \mathcal{P}_{\Pi(\varphi)} \mathbf{s}\| \geq \delta\},$$

where  $\mathcal{P}_{\Pi(\varphi)}$  is the orthogonal projection on the plane  $\Pi(\varphi) = \{\mathbf{u} \in \mathbb{R}^D : \langle \varphi, \mathbf{u} \rangle = 0\}$ .

Using the hyper spherical coordinate system developed earlier and denoting the angle  $\pi d_S(\mathbf{x}, \mathbf{s})$  by  $\theta$ , membership in  $\mathcal{V}_\delta^-$  can be expressed as

$$\varphi = \begin{cases} (r, \phi_1, \dots, \phi_{D-1}) \in \mathcal{V}_\delta^-, \\ \tan \phi_{D-1} \in [0, \tan \theta], \end{cases} \quad (\text{R1})$$

$$\Leftrightarrow \begin{cases} \sin \phi_1 \cdots \sin \phi_{D-2} |\sin \phi_{D-1}| \geq \delta, & (\text{R2}) \\ \sin \phi_1 \cdots \sin \phi_{D-2} |\sin(\phi_{D-1} - \theta)| \geq \delta. & (\text{R3}) \end{cases}$$

Indeed, requirement (R1) enforces  $\langle \varphi, \mathbf{x} \rangle \langle \varphi, \mathbf{s} \rangle \leq 0$ , while (R2) and (R3) are direct translations of the requirements that  $\|\mathbf{x} - \mathcal{P}_{\Pi(\varphi)} \mathbf{x}\| = |\langle \widehat{\varphi}, \mathbf{x} = \mathbf{e}_D \rangle| \geq \delta$  and  $\|\mathbf{s} - \mathcal{P}_{\Pi(\varphi)} \mathbf{s}\| = |\langle \widehat{\varphi}, \mathbf{s} = \cos \theta \mathbf{e}_D - \sin \theta \mathbf{e}_{D-1} \rangle| \geq \delta$ , with  $\widehat{\varphi} = \frac{1}{\|\varphi\|} \varphi$ .

We are now ready to integrate to find  $p_1 = \mu(\mathcal{V}_\delta^-)$ :

$$p_1 = \frac{1}{(2\pi)^{D/2}} \int_{\mathbb{R}_+} dr r^{D-1} e^{-r^2/2} \dots \\ \left[ \left( \int_0^\pi d\phi_1 \sin^{D-2} \phi_1 \right) \cdots \left( \int_0^\pi d\phi_{D-2} \sin \phi_{D-2} \right) \right] \dots \\ \left[ \int_{[0, \theta] \cup [\pi, \pi + \theta]} d\phi_{D-1} \chi_{g(\delta, \varphi)}(\phi_{D-1}) \chi_{g(\delta, \varphi)}(\phi_{D-1} - \theta) \right],$$

with  $\chi_\lambda(\phi) = 1$  if  $|\sin \phi| \geq \lambda$  and 0 else, for some  $\lambda \in [0, 1]$ , and  $g(\delta, \varphi) = \delta / (\sin \phi_1 \cdots \sin \phi_{D-2})$ .

However,

$$\int_{[0, \theta] \cup [\pi, \pi + \theta]} d\phi \chi_\lambda(\phi) \chi_\lambda(\phi - \theta) \\ = \max(2\theta - 4 \arcsin \lambda, 0),$$

and  $\max(2\theta - 4 \arcsin \lambda, 0) \geq 2\theta - 2\pi\lambda$ , since  $\lambda \leq \arcsin \lambda \leq \frac{\pi}{2} \lambda$  for any  $\lambda \in [0, 1]$ . Consequently,

$$\mu(\mathcal{V}_\delta^-) \geq \frac{1}{(2\pi)^{D/2}} \int_{\mathbb{R}_+} dr r^{D-1} e^{-r^2/2} \dots \\ \left[ \left( \int_0^\pi d\phi_1 \sin^{D-2} \phi_1 \right) \cdots \left( \int_0^\pi d\phi_{D-2} \sin \phi_{D-2} \right) \right] \cdots \\ \left( \frac{2\theta - 4 \arcsin \lambda}{\sin \phi_1 \cdots \sin \phi_{D-2}} \right) \\ = \frac{\theta}{\pi} - 2\pi\delta \frac{I_{D-3} I_{D-4} \cdots I_0}{(2I_0) I_1 \cdots I_{D-2}} = \frac{\theta}{\pi} - \frac{\pi \delta}{I_{D-2}},$$

with  $I_n := \int_0^\pi d\phi \sin^n \phi$  and knowing that  $(2\pi)^{D/2} = (2I_0) (I_1 \cdots I_{D-2}) \int_{\mathbb{R}_+} dr r^{D-1} e^{-r^2/2}$  since

$$\mu(\mathbb{R}^D) = 1 = (2\pi)^{-D/2} \int_{\mathbb{R}_+} dr r^{D-1} e^{-r^2/2} \dots \\ \left( \int_0^{2\pi} d\phi_{D-1} \right) (I_1 \cdots I_{D-2}).$$

Using the fact that  $I_n = \sqrt{\pi} \Gamma(\frac{n+1}{2}) / \Gamma(\frac{n}{2} + 1) \geq \sqrt{\pi} / \sqrt{\frac{n}{2} + \frac{1}{4}}$ , we obtain  $I_{D-2} \geq \frac{\sqrt{\pi}}{\sqrt{\frac{D}{2} - \frac{3}{4}}} \geq \sqrt{\frac{2\pi}{D}}$ , and thus

$$p_1 \geq d_S(\mathbf{x}, \mathbf{s}) - \sqrt{\frac{\pi}{2} D} \delta.$$

If we want a meaningful bound for  $p_1 \geq 0$ , then we must have  $d_S(\mathbf{x}, \mathbf{s}) \geq \sqrt{\frac{\pi}{2} D} \delta \geq \delta$ . Therefore, as soon as the lower bound is positive, the aforementioned condition  $d_S(\mathbf{x}, \mathbf{s}) \geq \delta$  always holds.

The lower bound for  $p_0$  is obtained similarly. It is straightforward to show that  $p_0 = \mu(\mathcal{V}_\delta^+)$ , with  $\mathcal{V}_\delta^+ = \{\varphi : \langle \varphi, \mathbf{x} \rangle \langle \varphi, \mathbf{s} \rangle > 0, \|\mathbf{x} - \mathcal{P}_{\Pi(\varphi)} \mathbf{x}\| \geq \delta, \|\mathbf{y} - \mathcal{P}_{\Pi(\varphi)} \mathbf{s}\| \geq \delta\}$ . Lower bounding  $\mu(\mathcal{V}_\delta^+)$  as for  $\mu(\mathcal{V}_\delta^-)$ , the only difference occurring with the integral on  $\phi_{D-2}$  given by

$$\int_{[\theta, \pi] \cup [\pi + \theta, 2\pi]} d\phi_{D-1} \chi_{g(\delta, \varphi)}(\phi_{D-1}) \chi_{g(\delta, \varphi)}(\phi_{D-1} - \theta) \\ = 2\pi - 2\theta - 4 \arcsin g(\delta, \varphi) \geq 2(\pi - \theta) - 2\pi g(\delta, \varphi).$$

Therefore, the lower bound of  $p_0$  amounts to change  $\theta \rightarrow \pi - \theta$  in the one of  $p_1$ , which provides the result. ■

*Proof of Lemma 7:* If  $\delta = 0$ , there is nothing to prove. Therefore  $\delta > 0$  and if  $\varphi^*$  belongs to either  $\mathcal{V}_\delta$  or  $\mathcal{W}_\delta$ , we must have  $\langle \varphi, \mathbf{x} \rangle \langle \varphi, \mathbf{s} \rangle < 0$ . It is also sufficient to work on the restriction of  $\mathcal{V}_\delta$  and  $\mathcal{W}_\delta$  to unit vectors.

(i)  $\mathcal{V}_\delta \subset \mathcal{W}_\delta$ : By contradiction, let us assume that  $\varphi^* \in \mathcal{V}_\delta$  but  $\varphi^* \notin \mathcal{W}_\delta$ . Without any loss of generality,  $\langle \varphi^*, \mathbf{x} \rangle > 0$  and  $\langle \varphi^*, \mathbf{s} \rangle < 0$ . Since  $\varphi^* \notin \mathcal{W}_\delta$ , there exist two vectors  $\mathbf{u}^* \in B_\delta^*(\mathbf{x})$  and  $\mathbf{v}^* \in B_\delta^*(\mathbf{s})$  such that  $\langle \varphi^*, \mathbf{u}^* \rangle \langle \varphi^*, \mathbf{v}^* \rangle > 0$ . If  $\langle \varphi^*, \mathbf{u}^* \rangle > 0$  and  $\langle \varphi^*, \mathbf{v}^* \rangle > 0$ , then, since  $\langle \varphi^*, \mathbf{s} \rangle < 0$  and by continuity of the inner product, there exist a  $\lambda \in (0, 1)$  such that  $\langle \varphi^*, \mathbf{s}(\lambda) \rangle = 0$  with  $\mathbf{s}(\lambda) = \mathbf{s} + \lambda(\mathbf{v}^* - \mathbf{s})$ . Therefore,  $\mathbf{s}(\lambda) \in \Pi(\varphi)$  and, by definition of the orthogonal projection,  $\|\mathbf{s} - \mathcal{P}_{\Pi(\varphi)} \mathbf{s}\| \leq \|\mathbf{s} - \mathbf{s}(\lambda)\| \leq \lambda \delta < \delta$  which is a contradiction. If  $\langle \varphi^*, \mathbf{u}^* \rangle < 0$  and  $\langle \varphi^*, \mathbf{v}^* \rangle < 0$ , we apply the same reasoning on  $\mathbf{x}$  and  $\mathbf{u}^*$ . Therefore,  $\mathcal{V}_\delta \subset \mathcal{W}_\delta$ .

(ii)  $\mathcal{W}_\delta \subset \mathcal{V}_\delta$ : If  $\varphi^* \in \mathcal{W}_\delta$  with  $\varphi^* \notin \mathcal{V}_\delta$ , we have either  $\|\mathbf{x} - \mathcal{P}_{\Pi(\varphi^*)} \mathbf{x}\| < \delta$  or  $\|\mathbf{s} - \mathcal{P}_{\Pi(\varphi^*)} \mathbf{s}\| < \delta$ . Let us say that  $\|\mathbf{x} - \mathcal{P}_{\Pi(\varphi^*)} \mathbf{x}\| < \delta$ . Then, for  $\mathbf{w} = \mathbf{x} + \delta(\mathcal{P}_{\Pi(\varphi^*)} \mathbf{x} - \mathbf{x}) / \|\mathcal{P}_{\Pi(\varphi^*)} \mathbf{x} - \mathbf{x}\| \in B_\delta^*(\mathbf{x})$ ,  $\langle \varphi^*, \mathbf{x} \rangle \langle \varphi^*, \mathbf{w} \rangle = (\langle \varphi^*, \mathbf{x} \rangle)^2 (1 - \delta / \|\mathcal{P}_{\Pi(\varphi^*)} \mathbf{x} - \mathbf{x}\|) + \delta \langle \varphi^*, \mathcal{P}_{\Pi(\varphi^*)} \mathbf{x} \rangle < 0$ . However,  $\varphi^* \in \mathcal{W}_\delta$  and  $\langle \varphi^*, \mathbf{x} \rangle \langle \varphi^*, \mathbf{s} \rangle < 0$ , leading to  $\langle \varphi^*, \mathbf{w} \rangle \langle \varphi^*, \mathbf{s} \rangle > 0$ , which is a contradiction. ■

APPENDIX E

THEOREM 3: GAUSSIAN MATRICES PROVIDE BeSES

The strategy for proving Theorem 3 will be to count the number of pairs of  $K$ -sparse signals that are Euclidean distance  $\delta$  apart. We will then apply the concentration results of Lemma 3 to demonstrate that the angles between these pairs are approximately preserved. We specifically proceed by focusing on a single  $K$ -dimensional subspace (intersected with the unit sphere) and then by applying a union bound to account for all possible subspaces.

Let  $T \subset [N]$  be an index set of size  $|T| = K$ ,  $\Sigma^*(T) = \{\mathbf{w} \in \mathbb{R}^N : \text{supp } \mathbf{w} \subset T, \|\mathbf{w}\|_2 = 1\}$  be the sphere of unit vectors with support  $T$ . We first use again the fact that the sphere  $\Sigma^*(T)$  can be  $\delta$ -covered by a finite set of points  $Q_{T,\delta}$ . That is, for any  $\mathbf{w} \in \Sigma^*(T)$ , there exists a  $\mathbf{q} \in Q_{T,\delta}$  such that  $\mathbf{w} \in B_\delta^*(\mathbf{q}) = B_\delta(\mathbf{q}) \cap \Sigma_T^* = \{\mathbf{w}' \in \Sigma_T^* : \|\mathbf{w}' - \mathbf{q}\|_2 \leq \delta\}$  [15]. Note that the size of  $Q_{T,\delta}$  is bounded by  $|Q_{T,\delta}| \leq C_\delta = (3/\delta)^K$ .

Let  $\Phi_T$  be the matrix formed by the columns of  $\Phi$  indexed by  $T$  and note that  $\Phi_T \mathbf{w} = \Phi \mathbf{w}$ . Given  $\epsilon' \geq 0$ , for all pairs of points  $\mathbf{p}, \mathbf{q} \in Q_{T,\delta}$ , we have

$$\begin{aligned} & \mathbb{P}\left(\forall \mathbf{u} \in B_\delta^*(\mathbf{p}), \forall \mathbf{v} \in B_\delta^*(\mathbf{q}), \right. \\ & \left. |d_H(A(\mathbf{u}), A(\mathbf{v})) - d_S(\mathbf{p}, \mathbf{q})| \leq \epsilon' + \sqrt{\frac{\pi}{2}} K \delta \right) \\ & \geq 1 - 2\left(\frac{3}{\delta}\right)^{2K} e^{-2\epsilon'^2 M}. \end{aligned} \quad (25)$$

This follows from Lemma 3 with  $D = K$ , since  $\Phi_T$  is a Gaussian matrix and by invoking the union bound, since there are  $\binom{C_\delta}{2} \leq C_\delta^2 = (3/\delta)^{2K}$  such pairs  $\mathbf{x}, \mathbf{s}$ .

The bound (25) can be extended to all possible index sets  $T$  of size  $K$  via the union bound. Specifically, for all  $T \subset [N]$  and all pairs of points  $\mathbf{p}, \mathbf{q} \in Q_{T,\delta}$ , we have now jointly

$$\begin{aligned} & \mathbb{P}\left(\forall \mathbf{u} \in B_\delta^*(\mathbf{p}), \forall \mathbf{v} \in B_\delta^*(\mathbf{q}), \right. \\ & \left. |d_H(A(\mathbf{u}), A(\mathbf{v})) - d_S(\mathbf{p}, \mathbf{q})| \leq \epsilon' + \sqrt{\frac{\pi}{2}} K \delta \right) \\ & \geq 1 - 2\left(\frac{eN}{K}\right)^K \left(\frac{3}{\delta}\right)^{2K} e^{-2\epsilon'^2 M} \end{aligned} \quad (26)$$

since there are no more than  $\binom{N}{K} \leq (eN/K)^K$  possible  $T$ .

We can reformulate this last result as follows. Let us take any pair of points on the sphere  $\mathbf{x}, \mathbf{s} \in S^{N-1}$  such that their joint support  $T = \text{supp}(\mathbf{x}) \cup \text{supp}(\mathbf{s})$  has a size  $|T| \leq K$ . We have obviously  $\mathbf{x}, \mathbf{s} \in \Sigma^*(T)$ . Taking the covering set  $Q_{T,\delta}$  defined for  $\Sigma^*(T)$ , there

exist two points  $\mathbf{p}, \mathbf{q} \in Q_{T,\delta}$  such that  $\mathbf{x} \in B_\delta^*(\mathbf{p})$  and  $\mathbf{s} \in B_\delta^*(\mathbf{q})$ . From (26), with a probability exceeding  $1 - 2\left(\frac{eN}{K}\right)^K \left(\frac{3}{\delta}\right)^{2K} e^{-2\epsilon'^2 M}$ , we have

$$|d_H(A(\mathbf{x}), A(\mathbf{s})) - d_S(\mathbf{p}, \mathbf{q})| \leq \epsilon' + \sqrt{\frac{\pi}{2}} K \delta. \quad (27)$$

To obtain our final bound, consider that  $\mathbf{x} \in B_\delta^*(\mathbf{p})$  implies that  $\pi d_S(\mathbf{x}, \mathbf{p}) \leq 2 \arcsin \delta/2 \leq \pi \delta/2$ , and  $d_S(\mathbf{s}, \mathbf{q})$  can be similarly bounded. Thus,  $d_S(\mathbf{x}, \mathbf{s}) \geq d_S(\mathbf{p}, \mathbf{q}) - \delta$  and  $d_S(\mathbf{x}, \mathbf{s}) \leq d_S(\mathbf{p}, \mathbf{q}) + \delta$ , and (27) becomes

$$|d_H(A(\mathbf{x}), A(\mathbf{s})) - d_S(\mathbf{x}, \mathbf{s})| \leq \epsilon' + (1 + \sqrt{\frac{\pi}{2}} K) \delta.$$

Let us define the probability of failure as  $2\left(\frac{eN}{K}\right)^K \left(\frac{3}{\delta}\right)^{2K} e^{-2\epsilon'^2 M} = \eta$ , where  $0 < \eta < 1$ , and set  $\epsilon' = (1 + \sqrt{\frac{\pi}{2}} K) \delta$  and  $2\epsilon' = \epsilon$ . Solving for  $M$ , we finally get that  $|d_H(A(\mathbf{x}), A(\mathbf{s})) - d_S(\mathbf{x}, \mathbf{s})| \leq \epsilon$  with a probability bigger than  $1 - \eta$  if

$$M \geq \frac{2}{\epsilon^2} \left( K \log\left(\frac{9eN}{K}\right) + 2K \log\left(\frac{2(1 + \sqrt{2\pi}K)}{\epsilon}\right) + \log\left(\frac{2}{\eta}\right) \right).$$

Since  $K \geq 1$ , we have that  $2(1 + \sqrt{2\pi}K) \leq 2(1 + \sqrt{2\pi})\sqrt{K} < 35\sqrt{K}/\sqrt{9e}$ , and thus the previous relation is satisfied if

$$\begin{aligned} M & \geq \frac{2}{\epsilon^2} \left( K \log\left(\frac{9eN}{K}\right) + 2K \log\left(\frac{35/\sqrt{9e}}{\epsilon} \sqrt{K}\right) + \log\left(\frac{2}{\eta}\right) \right), \\ & = \frac{2}{\epsilon^2} \left( K \log(N) + 2K \log\left(\frac{35}{\epsilon}\right) + \log\left(\frac{2}{\eta}\right) \right). \end{aligned}$$

APPENDIX F

LEMMA 4: STABILITY WITH MEASUREMENT NOISE

In Lemma 4, since  $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$ , each  $y_i = (\Phi \mathbf{x})_i$  follows a Gaussian distribution  $\mathcal{N}(0, \|\mathbf{x}\|_2^2)$ , and furthermore, since we have independent additive noise,  $z_i = y_i + n_i = (\Phi \mathbf{x})_i + n_i$  follows the Gaussian distribution  $\mathcal{N}(0, \|\mathbf{x}\|_2^2 + \sigma^2)$ .

We begin by bounding the probability that any noisy measurement  $z_i$  has a different sign than the original corresponding measurement  $y_i$ , i.e., we bound  $\tilde{p} := \mathbb{P}(z_i y_i < 0)$ . This quantity is interesting since  $M d_H(A_n(\mathbf{x}), A(\mathbf{x}))$  follows a Binomial distribution with  $M$  trials and probability of success  $\tilde{p}$  and thus we also have  $\mathbb{E}(d_H(A_n(\mathbf{x}), A(\mathbf{x}))) = \tilde{p}$ .

To solve for the bound, we compute

$$\begin{aligned} \tilde{p} & = \int_{\mathbb{R}} du \mathbb{P}(z_i y_i < 0 | y_i = u) f_{y_i}(u) \\ & = \int_{\mathbb{R}} du \mathbb{P}(u^2 + u n_i < 0) g(u; \|\mathbf{x}\|_2), \end{aligned}$$

with the pdf  $f_{y_i}(t) = g(t; \sigma') = \frac{1}{\sqrt{2\pi t}} \exp(-t^2/2\sigma'^2)$ . This leads to

$$\begin{aligned} \tilde{p} &= \int_0^\infty du \mathbb{P}(n_i < -u) g(u; \|\mathbf{x}\|_2) \\ &\quad + \int_{-\infty}^0 du \mathbb{P}(n_i > -u) g(u; \|\mathbf{x}\|_2) \\ &= \int_0^\infty du 2Q(u/\sigma) g(u; \|\mathbf{x}\|_2) \\ &\leq \int_0^\infty du e^{-\frac{u^2}{2\sigma^2}} g(u; \|\mathbf{x}\|_2) \\ &= \frac{1}{\sqrt{2\pi}\|\mathbf{x}\|_2} \int_0^\infty du \exp\left(-\frac{(\|\mathbf{x}\|_2^2 + \sigma^2)u^2}{2\sigma^2\|\mathbf{x}\|_2^2}\right) \\ &= \frac{1}{2} \frac{\sigma}{\sqrt{\|\mathbf{x}\|_2^2 + \sigma^2}}, \end{aligned}$$

where  $Q(u) = \int_u^\infty dt g(t; 1)$  denotes the tail integral of the standard Gaussian distribution which is bounded by  $Q(t) \leq \frac{1}{2}e^{-t^2/2}$  for  $t \geq 0$  (see for instance [69, Eq. (13.48)]).

Thus, we have  $\tilde{p} \leq e(\sigma, \|\mathbf{x}\|_2) = \frac{1}{2} \frac{\sigma}{\sqrt{\|\mathbf{x}\|_2^2 + \sigma^2}}$  and, by applying the Chernoff-Hoeffding inequality to the distribution of  $d_H(A_n(\mathbf{x}), A(\mathbf{x}))$ ,

$$\begin{aligned} \mathbb{P}[M d_H(A_n(\mathbf{x}), A(\mathbf{x})) > M e(\sigma, \|\mathbf{x}\|_2) + M\epsilon] \\ \leq \mathbb{P}[M d_H(A_n(\mathbf{x}), A(\mathbf{x})) > M \tilde{p} + M\epsilon] \\ \leq e^{-2M\epsilon^2}, \end{aligned}$$

which proves the lemma.

## APPENDIX G

### ASYMPTOTIC BOUND ON $\epsilon$ IN THEOREMS 2 AND 3

Both Theorem 2 and Theorem 3 provide guarantees on the worst-case error  $\epsilon$  of the form

$$\epsilon^n \leq \frac{1}{M} (\alpha K \log(N) + \beta \log(\frac{1}{\eta}) + \gamma K \log(\frac{1}{\epsilon})), \quad (28)$$

for some exponent  $n \in \{1, 2\}$ , and for given constants  $\alpha, \beta, \gamma > 0$ .

In this appendix we show that, considering  $0 < \eta < 1$  fixed, the relation (28) implies

$$\epsilon^n = O\left(\frac{K}{M} \log\left(\frac{MN}{K}\right)\right) \quad (29)$$

asymptotically in  $M/K$  and  $N$ . Notice that, up to a redefinition  $\epsilon^n \rightarrow \epsilon$  and  $\gamma/n \rightarrow \gamma$ , it is sufficient to prove the relation for  $n = 1$ . We also define  $\rho := \beta \log(1/\eta)$ .

First, we consider  $N$  fixed and show  $\epsilon = O\left(\frac{K}{M} \log\left(\frac{M}{K}\right)\right)$ . Let us assume this is not the case, i.e., for all  $c > 0$ , and all  $R_0 > 0$ , there exists a ratio  $M/K > R_0$  such that  $\epsilon > c(K/M) \log(M/K)$ . Therefore,

$$\log \frac{1}{\epsilon} < \log \frac{M}{K} - \log(c \log \frac{M}{K}) < \log \frac{M}{K} - \log(c \log R_0),$$

Thus (28) becomes

$$\epsilon \leq \frac{1}{M} (\alpha K \log N + \rho + \gamma K \log \frac{M}{K} - \gamma K \log(c \log R_0)).$$

Using  $\epsilon > c(K/M) \log(M/K)$ , this last inequality becomes

$$\begin{aligned} \alpha \log N + \frac{1}{K} \rho + \gamma \log \frac{M}{K} > \\ c \log \frac{M}{K} + \gamma \log(c \log R_0). \quad (30) \end{aligned}$$

For fixed  $N$  and  $\eta$ , and since we reasonably have  $K \geq 1$ , the parameters  $c$  and  $R_0$  can always be selected so that  $\gamma \log(c \log R_0) > \alpha \log N + \rho/K$ . In this case, (30) implies  $\gamma \log \frac{M}{K} > c \log \frac{M}{K}$ . Taking  $c > \gamma$ , which is still compatible with the selection of  $R_0$  and  $c$  above, leads to a contradiction. Thus  $\epsilon = O\left(\frac{K}{M} \log \frac{M}{K}\right)$  for fixed  $N$ .

Next, we assume  $N$  varies and  $R := M/K$  is fixed, and show that  $\epsilon^n = O((1/R) \log(RN))$ . We again restrict the analysis to  $n = 1$ . Now we assume for all  $N_0 > 0$  and all  $c > 0$ , there is a  $N > N_0$  such that  $\epsilon > (c/R) \log(RN)$ . This means that  $\log \frac{1}{\epsilon} < -\log((c/R) \log(RN))$  and (28) implies

$$\frac{c}{R} \log(RN) + \frac{\gamma}{R} \log\left(\frac{c}{R} \log(RN)\right) < \frac{\alpha}{R} \log N + \rho.$$

Since  $\alpha$  and  $\rho$  are fixed, selecting  $c > \max(\alpha, R\rho/\log R)$  and  $N_0$  such that  $\log(RN_0) > R/C$  leads to a contradiction and completes the proof.

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